(2+1)-dimensional solitons under perturbations

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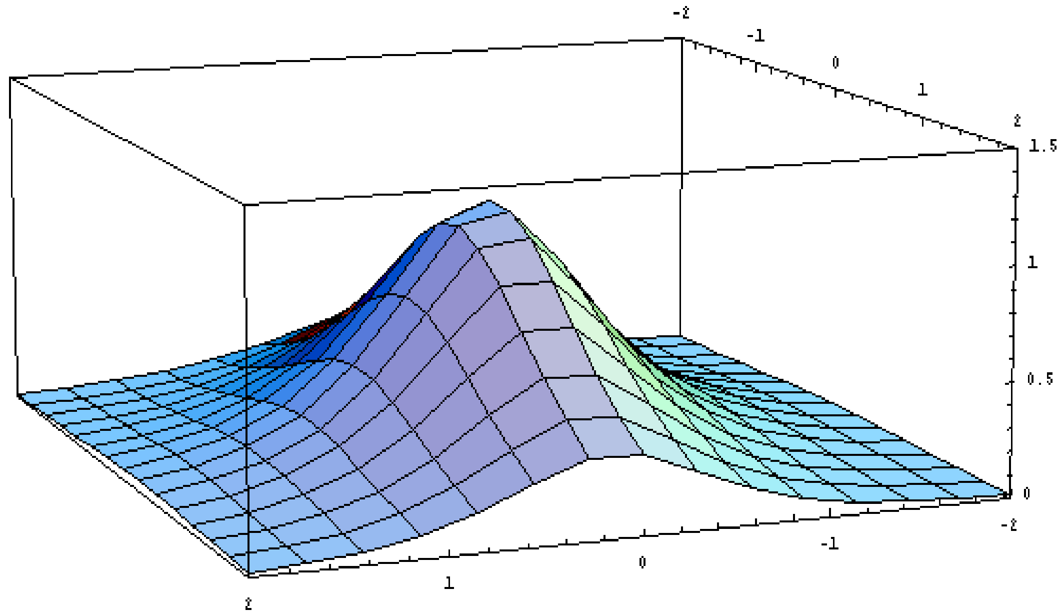
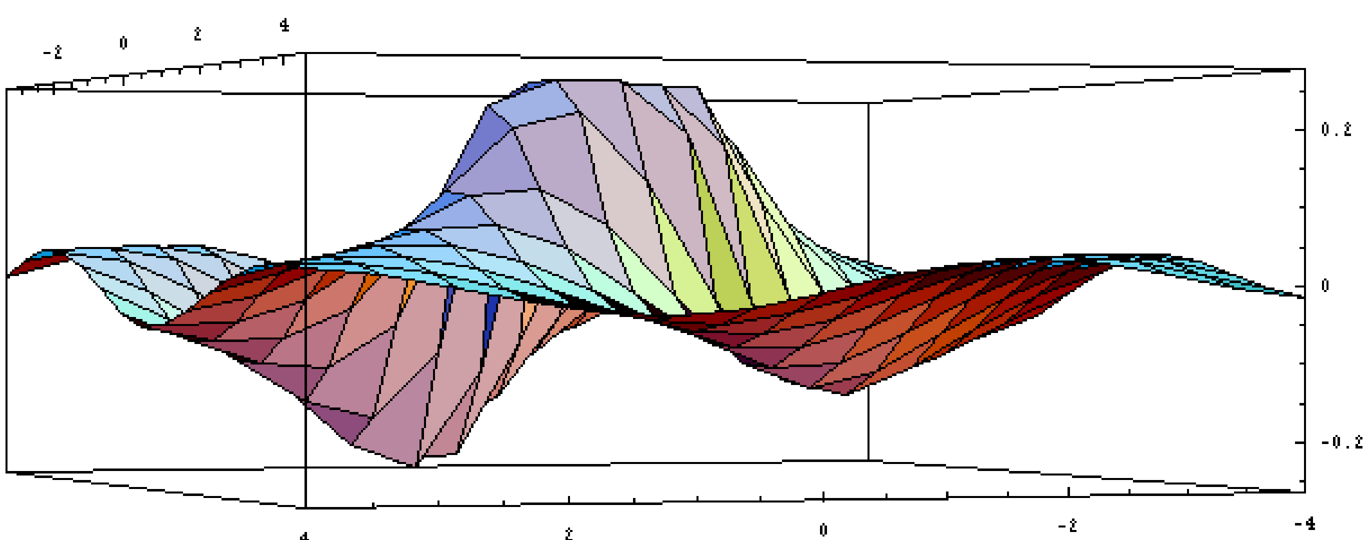
Two types of integrable Davey-Stewartson equations have two different types of (2+1)-dimensional solitons.

Soliton for the DS-1 equations was constructed in 1988 by Boiti, Leon, Martina and Pempinelli. This soliton exponentially decays at all spatial variables and its amplitude oscillates.



Almost in that time (1989) Arkadiev, Pogrebkov and Polivanov found a new nonsingular soliton for the DS-2 equation. Their solution decays algebraically with respect to spatial variables and also oscillates in time.



Perurbation of the DS-1 equations

$$i\partial_t Q + \frac{1}{2}(\partial_\xi^2 + \partial_\eta^2)Q + (G_1 + G_2)Q = \varepsilon i F,$$
$$\partial_\xi G_1 = -\frac{\sigma}{2}\partial_\eta |Q|^2, \quad \partial_\eta G_2 = -\frac{\sigma}{2}\partial_\xi |Q|^2. \tag{1}$$

Here ε is small positive parameter, $\sigma = \pm 1$ correspond to so called focusing or defocusing DS-1 equations.

The perturbations of the equations arise due to a small irregularity of bottom or by taking into account the next corrections in more realistic models for the liquid surface than was considered by Davey, Stewartsonand Djorjevic, Redekopp. For the first case the perturbation takes the form: $F \equiv AQ$. Here A is real constant and its sign corresponds to decreasing or increasing depth with respect to spatial variable ξ.

We start with the solution of DS-1 (as $\varepsilon = 0$) constructed by Boiti, Leon, Martina and Pempinelli:

$$q(\xi,\eta,t;\rho) = \frac{\rho\lambda\mu \exp(it(\lambda^2+\mu^2))}{2\cosh(\mu\xi)\cosh(\lambda\eta)} \times$$
$$\frac{1}{(1-\sigma|\rho|^2\frac{\mu\lambda}{16}(1+\tanh(\lambda\eta))(1+\tanh(\mu\xi)))}, \tag{2}$$

$$g_1|_{\xi\to-\infty} \equiv \frac{\lambda^2}{2\cosh^2(\lambda\eta)}, \quad g_2|_{\eta\to-\infty} \equiv \frac{\mu^2}{2\cosh^2(\mu\xi)}.$$

where λ, μ are positive constants defined by boundary conditions as $\eta \to -\infty$ and $\xi \to -\infty$; ρ is free complex parameter.

Let us seek an asymptotic solution in the form:

$$
\begin{aligned}
Q(\xi,\eta,t,\varepsilon) &= q(\xi,\eta,t;\tau) + \varepsilon U(\xi,\eta,t,\tau),\\
G_1(\xi,\eta,t,\varepsilon) &= g_1(\xi,\eta,t,\tau) + \varepsilon V_1(\xi,\eta,t,\tau),\\
G_2(\xi,\eta,t,\varepsilon) &= g_2(\xi,\eta,t,\tau) + \varepsilon V_2(\xi,\eta,t,\tau),
\end{aligned} \tag{3}
$$

Here main terms are solution of non-perturbed equations, but we suppose dependence of slow time $\tau = \varepsilon t$.

To construct the solution we must solve linearized DS-1 equations on the dromion as a background:

$$
i\partial_t U + (\partial_\xi^2 + \partial_\eta^2)U + (g_1 + g_2)U + (V_1 + V_2)q = iF
$$

$$
\partial_\xi V_1 = -\frac{\sigma}{2}\partial_\eta(q\bar{U} + \bar{q}U), \quad \partial_\eta V_2 = \frac{-\sigma}{2}\partial_\xi(q\bar{U} + \bar{q}U).
$$

In the inverse scattering transform one use the matrix solution of the Dirac system to solve the DS-1 equations Nizhnik, Fokas and Ablowitz, Fokas and Santini:

$$
\begin{pmatrix} \partial_\xi & 0 \\ 0 & \partial_\eta \end{pmatrix}\psi = -\frac{1}{2}\begin{pmatrix} 0 & q \\ \sigma\bar{q} & 0 \end{pmatrix}\psi. \tag{4}
$$

Let ψ^+ and ψ^- be the matrix solutions of the Goursat problem (following by Fokas and Santini):

$$
\begin{aligned}
&\psi_{11}^+|_{\xi\to-\infty} = \exp(ik\eta), && \psi_{12}^+|_{\xi\to-\infty} = 0,\\
&\psi_{21}^+|_{\eta\to\infty} = 0, && \psi_{22}^+|_{\eta\to-\infty} = \exp(-ik\xi);\\
&\psi_{11}^-|_{\xi\to-\infty} = \exp(ik\eta), && \psi_{12}^-|_{\xi\to\infty} = 0,\\
&\psi_{21}^-|_{\eta\to-\infty} = 0, && \psi_{22}^-|_{\eta\to-\infty} = \exp(-ik\xi).
\end{aligned} \tag{5}
$$

Denote by $\psi_{(j)}^+$, $j = 1, 2$, the columns of the matrix ψ^+.

Next two bilinear forms are analogs of the direct and inverse Fourier transforms:

$$(\chi, \mu)_f = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi d\eta (\chi_1 \mu_1 \, \sigma \bar{f} + \chi_2 \mu_2 \, f);$$

here χ_i and μ_i are the elements of the columns χ and μ.

$$\langle \chi, \mu \rangle_s = \int_{\mathbb{R}^2} dk dl (\chi^1(l) \mu^1(k) \, \sigma \bar{s}(k, l) + \chi^2(l) \mu^2(k) \, s(k, l)),$$

where χ^j is the element of the row χ.
Denote by $\varphi^{(j)}$, $j = 1, 2$, the row conjugated to $\psi^{(j)} = [\psi_{j1}^-, \psi_{j2}^+]$ with respect to the second bilinear form.

Theorem 1 (On decomposition) *Let Q be such that $\partial^\alpha Q \in L_1 \cap C^1$ for $|\alpha| \leq 3$, if a function f is such that $\partial^\alpha f(\xi, \eta) \in L_1 \cap C^1$ for $|\alpha| \leq 4$, then it may be represented in the form:*

$$f = \frac{-1}{\pi} \langle \psi^{(1)}(\xi, \eta, l), \varphi^{(1)}(\xi, \eta, k) \rangle_{\widehat{f}},$$

where

$$\widehat{f} = \frac{1}{4\pi} (\psi_{(1)}^+(\xi, \eta, k), \phi_{(1)}(\xi, \eta, l))_f.$$

Theorem 2 (Evolution of Fourier coefficients) *Let a solution of the first of the linearized DS-1 equations be smooth and integrable function U with respect to ξ and η, where $\partial^\alpha U \in L_1 \cap C^1$ and $\partial^\alpha F \in L_1 \cap C^1$, for $|\alpha| \leq 4$ and $t \in [0, T_0]$. Then*

$$\partial_t \widehat{U} = i(k^2 + l^2)\widehat{U} + \int_{-\infty}^{\infty} dk' \widehat{U}(k - k', l, t) \chi(k') + \int_{-\infty}^{\infty} dl' \widehat{U}(k, l - l', t) \kappa(l') + \widehat{F}. \quad (6)$$

Denote by $\gamma(\tau) = 1 - \sigma\frac{\mu\lambda}{4}|\rho(\tau)|^2$ and $\gamma_0 = \gamma(0)$. The final result is given by

Theorem 3 (Modulation of the soliton parameter) *If*

$$\gamma(\tau) = \gamma_0^{\exp(2A\tau)}, \quad Arg(\rho(\tau)) \equiv const,$$

where $\gamma_0 > 1$ at $\sigma = -1$ and $0 < \gamma_0 < 1$ at $\sigma = 1$, then the asymptotic solution (3) with respect to mod$(O(\varepsilon^2))$ is useful uniformly over $t = O(\varepsilon^{-1})$.

Corollary 1 (Very long times) *When the time is larger than ε^{-1}, namely, $t \ll \varepsilon^{-1}\log(\log(\varepsilon^{-1}))$, the formulas (3) are asymptotic solution of (1) with respect to mod$(o(1))$ only.*

Conjecture 1 (On a singularity) *The asymptotic analysis given here is valid for the solutions (2) without the singularities. It means if $\sigma = 1$, then $\frac{\mu\lambda}{4}|\rho(\tau)|^2 < 1$. If the coefficient of the perturbation $A > 0$, then $|\rho|$ increases with respect to slow time. It allows to say that the singularity may appear in the leading term of the asymptotics as $\tau \to \infty$. However we can't say this rigorously for perturbed dromion in our situation, because our asymptotics is usable only when $\tau \ll \log(\log(\varepsilon^{-1}))$. Generally the appearance of the singularities in the solution of nonintegrable cases of the Davey-Stewartson equations is known phenomenon Papanicolaou, C.Sulem, P.L.Sulem and Wang (1994).*

Perturbed solution of the DS-2 equations

Now I shall review results by R. Gadyl'shin and O. Kiselev (1996-99) concerning of soliton perturbation for the DS-2 equations:

$$i\partial_t q + 2(\partial_z^2 + \partial_{\bar z}^2)q + (g+\bar g)q = 0,$$
$$\partial_{\bar z} g = \partial_z |q|^2.$$

The nonsingular soliton obtained by Arkadiev, Pogrebkov and Polivanov (1989) has the form:

$$q(z,t) = \frac{2\bar\nu \exp(-it(k_0^2 + \bar k_0^2) + k_0 z - \overline{k_0 z})}{|z + 4ik_0 t + \mu|^2 + |\nu|^2}.$$

We have studied the solution of DS-2 with perturbed initial condition:

$$q_\varepsilon(z,0) = q(z,0) + \varepsilon q_1(z), \quad q_1(z) \in \mathcal{C}_0^\infty.$$

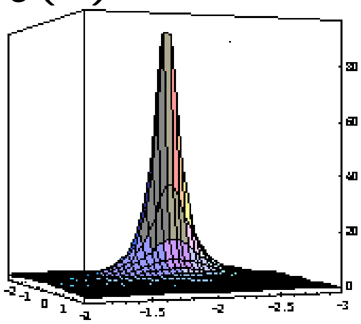
The DS-2 equation are associated with scattering problem for the Dirac equation Fokas, Ablowitz (1984) :

$$\begin{pmatrix} \partial_{\bar z} & 0 \\ 0 & \partial_z \end{pmatrix} \phi = \frac{1}{2} \begin{pmatrix} 0 & q(z,t) \\ \overline{-q(z,t)} & 0 \end{pmatrix} \phi,$$
$$\begin{pmatrix} \exp(kz) & 0 \\ 0 & \exp(\overline{kz}) \end{pmatrix} \phi(k,z)|_{|z|\to\infty} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have constructed an asymptotic solution for the scattering problem with perturbed potential and found that the scattering data had non-soliton structure:

$$b_\varepsilon(k) = \frac{-i}{4\pi} \int_{\mathbb{C}} dz \wedge d\bar z q(z) \phi_{22} \exp(-kz),$$

$$b_\varepsilon(k) \sim \varepsilon^{-1} B_{-1}\left(\frac{k-k_0}{}\right) + B_0\left(\frac{k-k_0}{\varepsilon}\right) \text{ when} |k-k_0| < 2\varepsilon^{1/2},$$



$$_{-1}(\kappa) = -\frac{\overline{Q_1}}{|Q_1|^2 + |Q_2+\kappa|^2},$$
$$Q_{1,2} = \text{const} \neq 0.$$

$$b_1(k) \quad \text{when } |k-k_0| > \varepsilon^{1/2}.$$

Next step is solving of $\bar{D}$-problem for the perturbed scattering data:

$$\begin{pmatrix} \partial_{\overline{k}} & 0 \\ 0 & \partial_k \end{pmatrix} \phi^T = \begin{pmatrix} 0 & \kappa \bar{b}_\varepsilon(k,t) \\ b_\varepsilon(k,t) & 0 \end{pmatrix} \phi^T,$$

$$\begin{pmatrix} \exp(-kz) & 0 \\ 0 & \exp(\overline{-kz}) \end{pmatrix} \phi^T(k,z)|_{|z|\to\infty} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We succeed in resolving this problem asymptotically and constructed the asymptotic solution as an asymptotics as $\varepsilon \to 0$ of the integral:

$$q_\varepsilon(z,t) = \frac{-i}{\pi}\int_{\mathbb{C}} dp \wedge d\overline{p}\, b_\varepsilon(p) \exp(2it(p^2+\overline{p}^2) - \overline{pz})\phi_{11}(p,z,t).$$

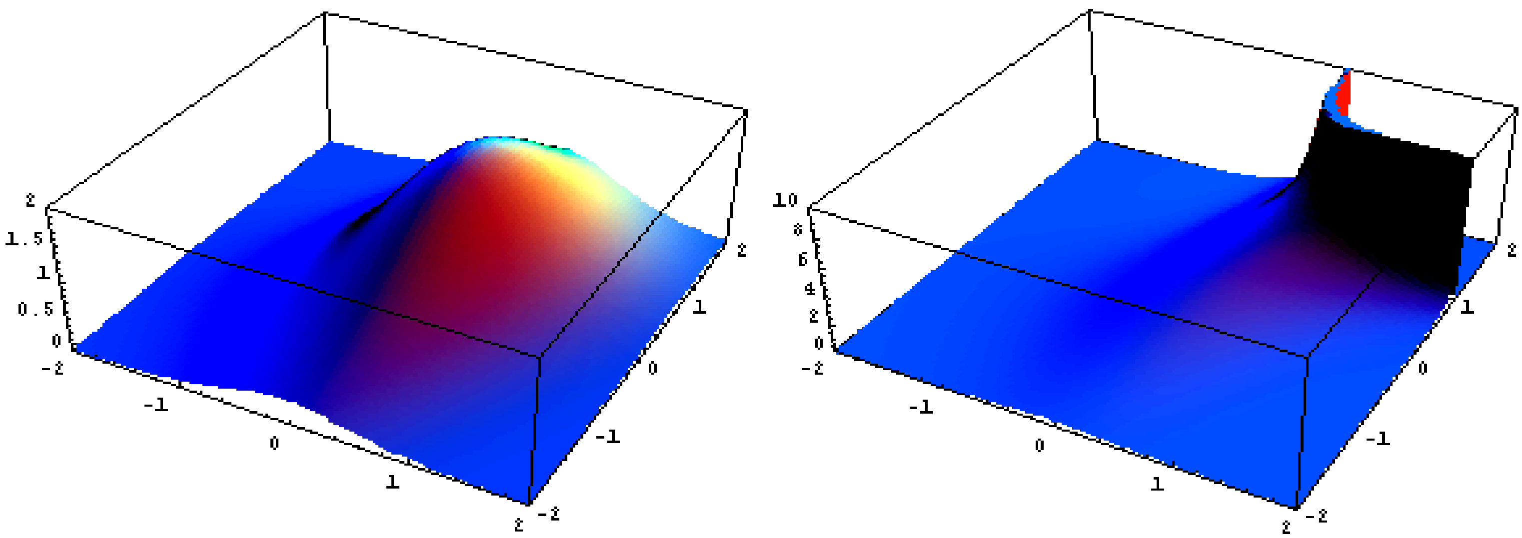
Proposition 1 (DS-2 soliton perturbation) *The perturbation of soliton in an initial data leads to non-soliton structure of the scattering data, but the solution concerns a soliton-like shape with modulated parameter $\mu_\varepsilon = \mu_0 + \varepsilon 2t\pi Q_2$ up to $t = O(\varepsilon^1)$.*

Over more large times $t = O(\varepsilon^{-1-\gamma})$, $\gamma > 0$ the asymptotic solution disperses:

$$q_\varepsilon \sim (t\varepsilon)^{-1} B_{-1}\left(\frac{iz}{4t}\right) \exp\left(\frac{i(z^2+\bar{z}^2)}{8t}\right).$$

Conclusions

The asymptotic analysis shows that a singularity may arise in soliton-like solution of the perturbed focusing DS-1 equations as $t = O(\varepsilon^{-1} \log(\log(\varepsilon^{-1})))$.



The solution of the DS-2 equations with perturbed soliton (lump) as the initial data disperses as $t \to \infty$.