Asymptotic description of separatrix crossing near a saddle-center point*

http://arxiv.org/abs/math/0105011

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*This work was suported by FRBR (00-01-00663, 00-15-96038) and INTAS (99-1068)

We consider a formal asymptotic solution of the main resonance equation:

$$i\varepsilon U' + (|U|^2 - t)U = 1, \quad 0 < \varepsilon \ll 1. \tag{1}$$

This equation defines a behaviour of an nonlinear system with cubic resonance and therefore it is one of major equations for nonlinear studies. The equation has been written in the form (1) which means the solution depend on fast (t) and slow (εt) typical scales.

We investigate a saddle-center bifurcation for the slowly varying equilibriums of this equation and construct a matching asymptotic solution uniformly as $\varepsilon \to 0$ before, inside and after the bifurcation layer.

This problem may be considered as a separatrix crossing in confluent point. The passing through a separatrix of the second order equations in a general position was considered by A.V. Timofeev in 1979, A.I. Neishtadt in 1986. The separatrix crossing for the second order equations in the confluent point was considered in a preliminary fashion by R.Haberman in 1979 and D.C.Diminnie and R.Haberman in 2000 in more detail. The asymptotic solution crossing the separatrix in the confluent point was constructed by O.M.Kiselev in 1999,2001 for the Painlevé-2 equation.

Using our approach one can construct the uniform asymptotic solution crossing the separatrix in the confluent point for second order equation in general case.

Algebraic analysis

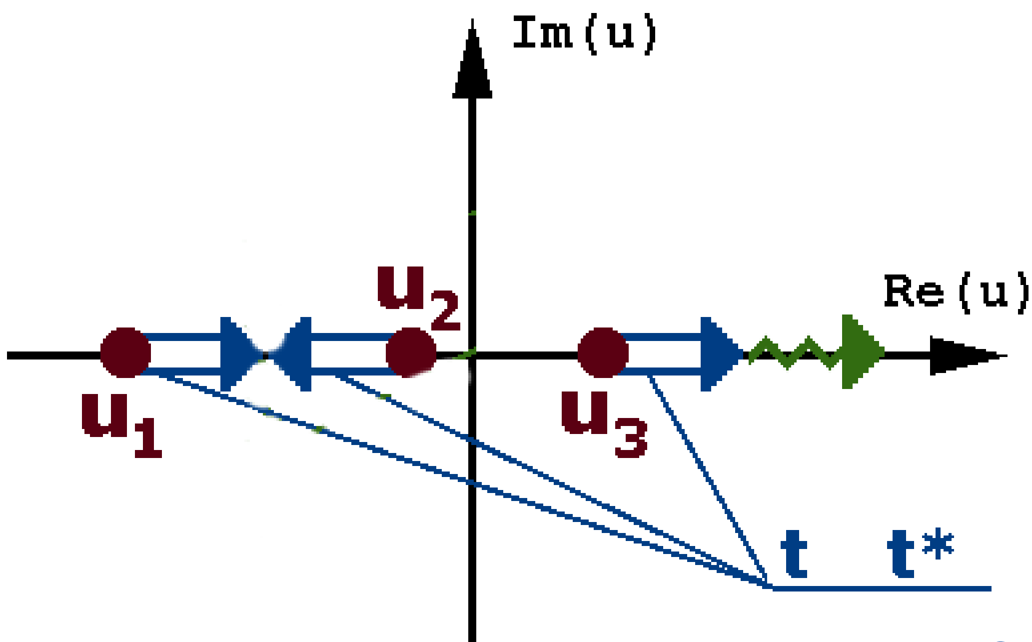
Let us seek an asymptotic solution of (1) in the form of the formal asymptotic series:

$$U(t;\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \overset{n}{U}(t). \qquad (2)$$

After a natural supposition about boundedness of derivative in the equation (1) we obtain the nonlinear equation for the main term of the asymptotic expansion:

$$|\overset{0}{U}|^2 \overset{0}{U} - t\overset{0}{U} = 1. \qquad (3)$$

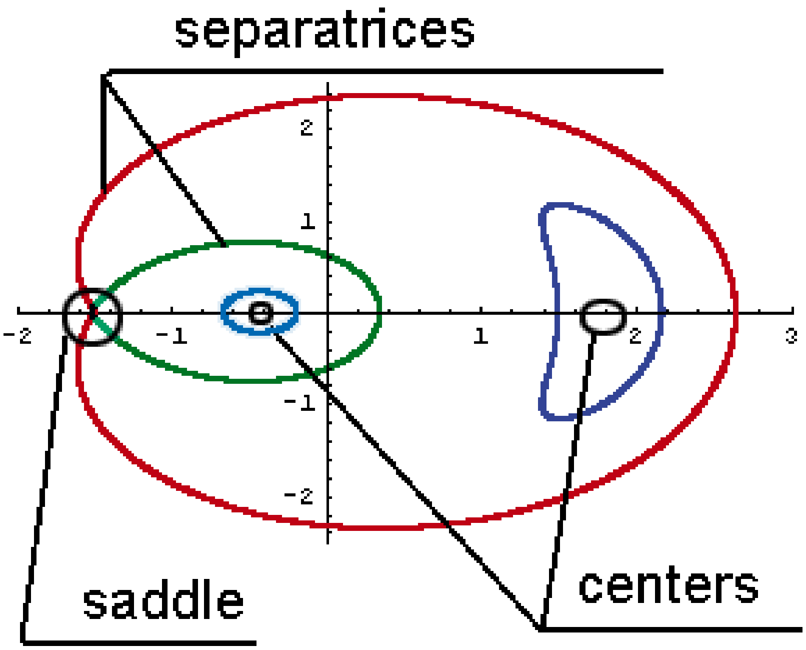
The number of the roots of this algebraic equation depends on a parameter t. There exist a value of the parameter t equals to $t_* = 3(1/2)^{2/3}$ so that the equation (3) has three real roots at $t > t_*$. At $t = t_*$ there is one simple root and one multiple root $U_* = -(1/2)^{1/3}$. At $t < t_*$ the equation (3) has the alone root.



What happened with the asymptotic solution of the equation (1) when two roots of the equation (3) coalesce?

Qualitative analysis

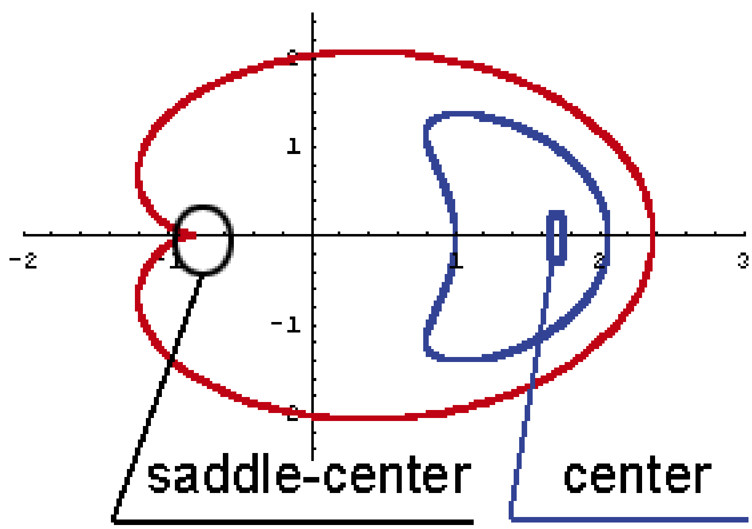
To obtain an answer on the precision question one can consider an autonomous equation with a "frozen" coefficient T:

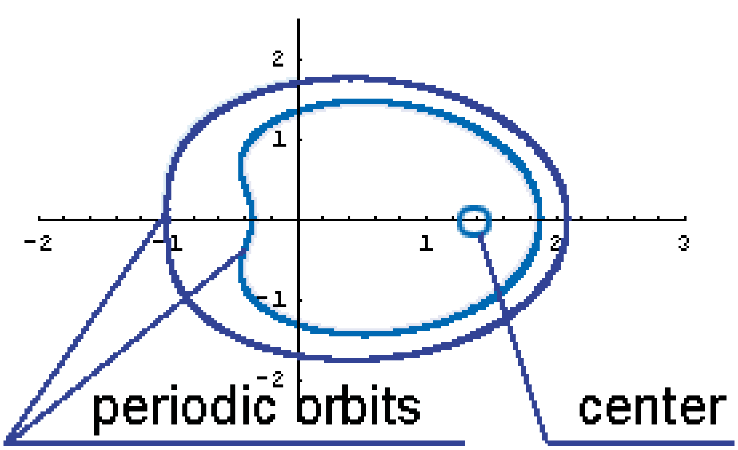


$$iV' + (|V|^2 - T)V = 1.$$

This equation has three equilibrium positions $T > t_*$. There are $U_1 < U_2 < U_3$, where U_1 is a saddle, U_2 and U_3 are centers.

At $T = t_*$ the saddle-node bifurcation takes place and there exist center U_3 and confluent saddle-center point U_*.





When $T < t_*$ there exists along center U_3.

Statement of the problem

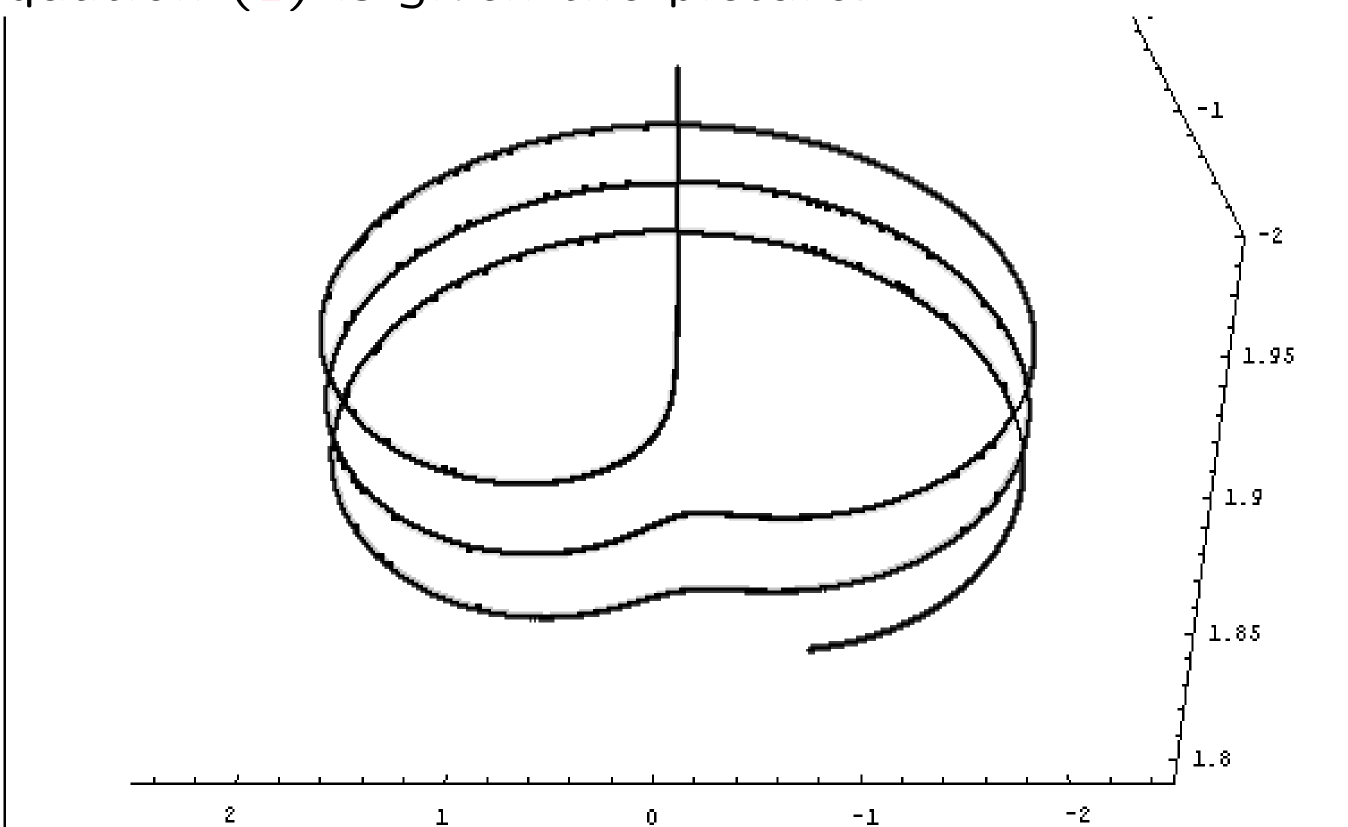
We will construct the formal asymptotic solution of the equation (1) in the interval $t \in [t_* - C, t_* + C]$ where $C = \text{const} > 0$ uniform on ε . We suppose that the solution in the domain $t > t_*$ has the form

$$U(t,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \overset{n}{U}(t), \quad \text{where } \overset{0}{U}(t) = U_2(t)$$

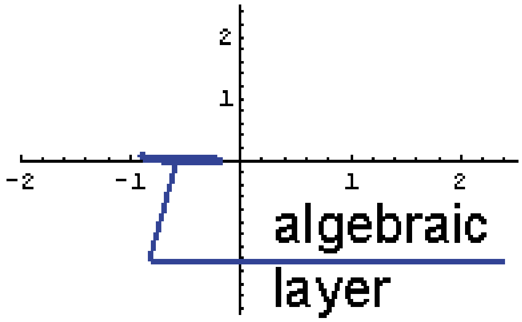
- The qualitative analysis shows that this asymptotic solution oscillates when $t < t_*$. Our problem is to study the transition layer between the nonoscillating asymptotics when $t > t_*$ and the oscillating asymptotics when $t < t_*$.

Numeric evaluations

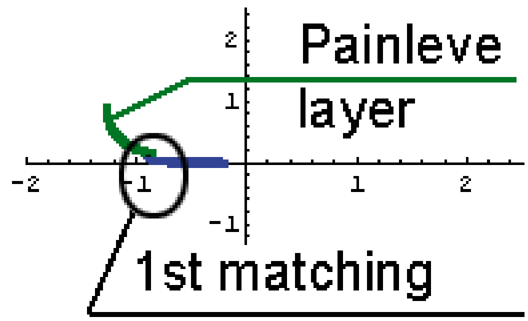
The numeric evaluations for the special solution of the equation (1) is given the picture:



Asymptotic analysis



In the domain $(t-t_*)\varepsilon^{-4/5} \gg 1$ the asymptotics has the form: $U(t,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \overset{n}{U}(t)$. Here $\overset{0}{U}(t) = U_2(t)$ and corrections $\overset{n}{U}(t)$ are algebraic functions of t. [R.Haberman, 1979]



In the domain $|t-t_*| \ll 1$ the asymptotics is defined by four various expansions of different types. First of them is:

$$U(t,\varepsilon) = U_* + \varepsilon^{2/5} \sum_{n=0}^{\infty} \varepsilon^{2n/5}\Big(\overset{n}{\alpha}(\tau) + i\varepsilon^{1/5} \overset{n}{\beta}(\tau)\Big), \quad (4)$$

where $\tau = (t-t_*)\varepsilon^{-4/5}$. The leader term $\overset{0}{\alpha}(\tau)$ is a special solution of the Painlevé-1 equation [R.Haberman, 1979]:

$$\overset{0}{\alpha}{}'' - 3\,\overset{0}{\alpha}{}^2 + \tau = 0,$$

with the given asymptotics as $\tau \to \infty$:

$$\overset{0}{\alpha}(\tau) = \sum_{n\ge 0} \alpha_n \tau^{-\frac{(5n-1)}{2}}, \quad \text{where } \alpha_0 = \frac{1}{\sqrt{3}},\ \alpha_1 = \frac{1}{24}.$$

In the domain $\tau > -\infty$ this solution has poles on the real axis of τ. Denote the largest of them by τ_0. The asymptotics (4) is valid as $(\tau-\tau_0)\varepsilon^{-1/5} \gg 1$.

In the neighborhood of $\tau = \tau_0$ the coefficients of the asymptotic expansion depend on one more fast time scale $\theta = (\tau - \tau_0)\varepsilon^{-1/5}$. Denote by

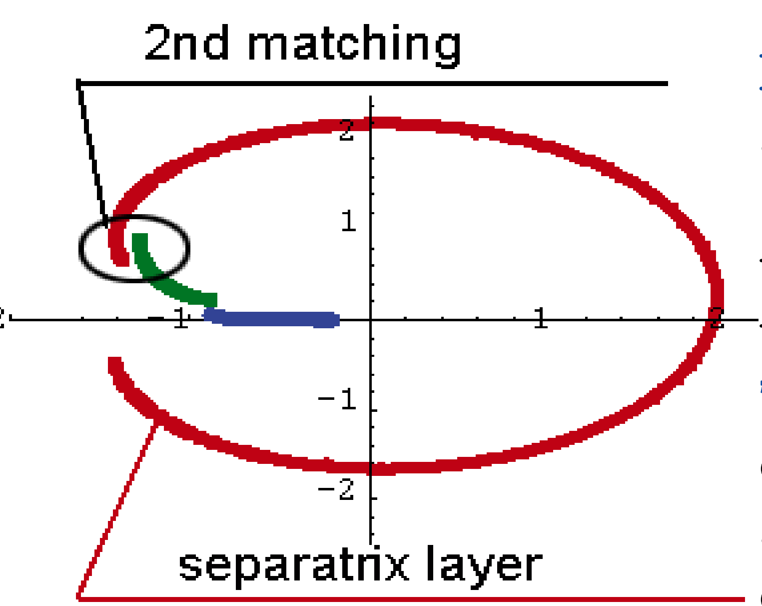
$$\theta_0 = \theta + \sum_{n=1}^{\infty} \varepsilon^{n/5} \overset{n}{\theta}_0,$$

where $\overset{n}{\theta}_0 = \text{const}$. Then in the domain $-\varepsilon^{-1/5} \ll \theta_0 \ll \varepsilon^{-1/10}$ the formal asymptotic solution has the form [Kiselev,1999,2001]:

$$U(t,\varepsilon) = U_* + \overset{0}{w}(\theta_0) + \varepsilon^{4/5} \sum_{n=1}^{\infty} \varepsilon^{(n-1)/5} \overset{n}{w}(\theta_0).$$

The main term of asymptotics $\overset{0}{w}(\theta_0)$ is the separatrix solution of the autonomous equation [R.Haberman, 1979]:

$$i \overset{0}{w}{}' + U_* \left(2|\overset{0}{w}|^2 + \overset{0}{w}{}^2\right) + U_*^2 \left(\overset{0}{w}{}^* - \overset{0}{w}\right) + |\overset{0}{w}|^2 \overset{0}{w} = 0, \quad (5)$$



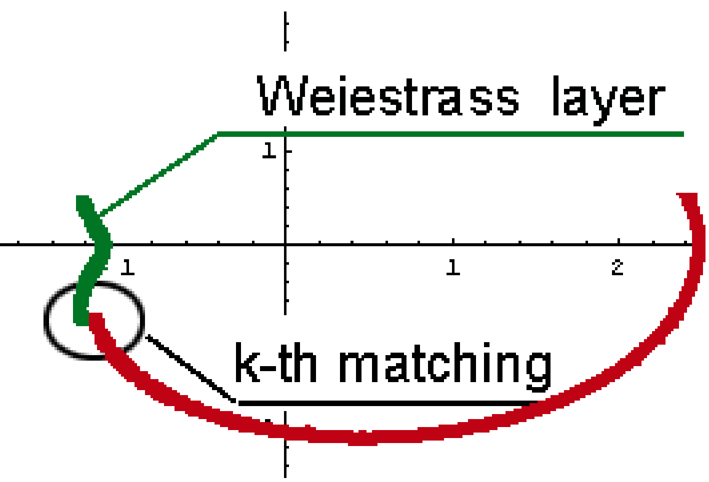
namely: $\overset{0}{w}(\theta_0) = \frac{-2}{(\theta_0 - iU_*)^2}$. In the domain $-\theta_0 \gg 1$ the asymptotic solution is defined by a sequence of two alternating asymptotics. Let us call them by "intermediate" and "separatrix" asymptotics. To obtain the intermediate asymptotics let us introduce one more slow variable:

$$T_k = \theta_{k-1}\varepsilon^{1/6}, k = 1, 2, \dots .$$

An asymptotic solution in the intermediate domain for not too large values $k \ll \varepsilon^{-1/7}$ has the form: $U(t,\varepsilon) = U_* + \varepsilon^{1/3}\sum_{n=0}^{\infty}\varepsilon^{i/30}\Big(\overset{n}{A}_k + i\varepsilon^{1/6}\overset{n}{B}_k\Big)$. The leader term satisfies to the equation [Diminnie& Haberman, 2000]:

$$\overset{0}{A}{}_k'' + 3\overset{0}{A}{}_k^2 = 0$$

and can be expressed by the Weierstrass $\wp$-function: $\overset{0}{A}_k = -2\wp(T_k; 0, g_3(k)),\ g_3(k) = \frac{1}{56}\Big(g_3(k-1) + \pi/2\Big)$.



Here $g_3(0) = \frac{a_4}{56}$, a_4 is the coefficient as $(\tau - \tau_0)^4$ in the Laurent expansion of $\overset{0}{\alpha}(\tau)$. The intermediate expansion with the leader term is valid in the domain between two poles $T_k = 0$ and $T_k = \Omega_k$ of the $\wp$-function: $-\varepsilon^{-1/6}T_k \gg 1,\ \varepsilon^{-2/15}(T_k + \Omega_k) \gg 1$.

At the large values of k the intermediate asymptotics are constructed in the form [Glebov& Kiselev, 2001] $U(t,\varepsilon) = U_* + \varepsilon^{1/3}\sum_{n=0}^{\infty}\varepsilon^{n/6}\Big(\overset{5n}{A}_k + i\varepsilon^{1/6}\overset{5n}{B}_k\Big)$.

The main term satisfies: $\overset{0}{A}{}_k'' + 3\overset{0}{A}{}_k^2 = \lambda_k$, where

$$\lambda_k(\varepsilon) = \varepsilon^{1/6}\left(\sum_{j=1}^{k}\Omega_j + \sum_{n=1}^{\infty}\varepsilon^{(n-1)/30}\sum_{j=1}^{k}\overset{n}{x}{}_j^{+}\right).$$

The main term of the asymptotics is:

$$\overset{0}{A}_k(T_k) = -2\wp(T_k, \lambda_k/2, g_3(k,\varepsilon)),$$

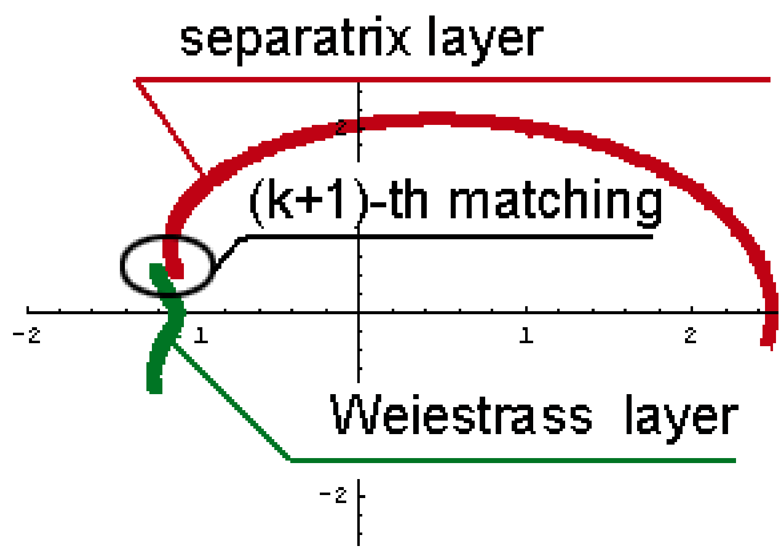
where $g_3(k,\varepsilon) = \overset{0}{g}_3(k) + \sum_{n=1}^{\infty} \varepsilon^{n/30} \overset{n}{g}_3(k)$.
The intermediate expansion with the leader term is valid in the domain between the poles of the Weierstrass function as

$$-\varepsilon^{-1/6} T_k \gg 1, \quad \varepsilon^{-2/15}(T_k + \Omega_k) \gg 1.$$

The separatrix expansions are valid in a small neighborhood of the Weierstrass function poles. Denote:
$\theta_k = \left(T_k + \Omega_k - \frac{1}{4}\sum_{n=1}^{\infty} \varepsilon^{n/30} \overset{n}{x}_k + \right)\varepsilon^{-1/6}, \quad k = 1, 2, \dots.$
When $|\theta_k|\varepsilon^{1/6} \ll 1$ the formal asymptotic solution of equation (1) has the form [Glebov& Kiselev, 2001]:

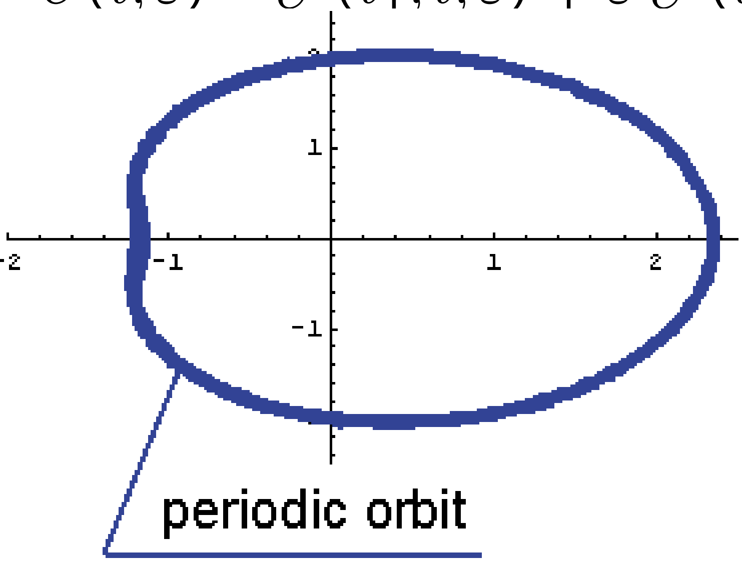
$$U(t,\varepsilon) = U_* + \overset{0}{W}(\theta_k) + \varepsilon^{4/5} \sum_{n=1}^{\infty} \varepsilon^{(n-1)/30} \overset{n}{W}(\theta_k).$$

The leader term of the asymptotics $\overset{0}{W}(\theta_k)$ is a separatrix solution of the autonomous equation (5):
$\overset{0}{W}(\theta_k) = \frac{-2}{(\theta_k - iU_*)^2}$.
The sequence of the alternating intermediate expansions and separatrix asymptotics is valid as $\varepsilon^{-1/6}(t_* - t) \ll 1$.



In the domain $(t_* - t)\varepsilon^{-2/3} \gg 1$ the asymptotic solution breaks into oscillation. The amplitude of the stimulated oscillations in the solution of (1) oscillates fast. The form of the solution is:

$U(t,\varepsilon) = \overset{0}{U}(t_1,t,\varepsilon) + \varepsilon \overset{1}{U}(t_1,t,\varepsilon) + \varepsilon^2 \overset{2}{U}(t_1,t,\varepsilon)$,



where t_1 is a new fast variable $t_1 = S(t)/\varepsilon + \phi(t)$. The main term of the symptotics $\overset{0}{U}$ lies on the curve $\Gamma(t)$: $\frac{1}{2}|y|^4 - t|y|^2 - (y+\bar{y}) = E(t)$, and satisfies to the Cauchy problem for the equation

$$iS'\partial_{t_1}\overset{0}{U} + (|\overset{0}{U}|^2 - t)\overset{0}{U} = 1,$$

with an initial condition $\overset{0}{U}|_{t_1=0} = u_0$, such, that $\mathrm{Im}(u_0) = 0$, $\mathrm{Re}(u_0) = \min_{y\in\Gamma(t)}\Big(\mathrm{Re}(y)\Big)$. The function $S(t)$ is a solution for the Cauchy problem

$$iS'\int_{\Gamma(t)} \frac{dy}{\sqrt{3y^3 + (2E+t^2)y^2 + 2ty + 1}} = T, S|_{t=0} = 0,$$

where $T = \mathrm{const} > 0$. The function $E(t)$ is the solution of the transcendental equation [Kuzmak, 1959]:

$$i\int_{\Gamma(t)} u^* du = \pi$$

The phase shift ϕ is defined by initial problems for the equation[Bourland& Haberman, 1988]:

$$\frac{\phi'}{\partial_E S}\partial_E I = \phi_1 = \mathrm{const}, \ \ \phi(t_*) = \phi_0.$$

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