Slow passage through resonance
for a weakly nonlinear dispersive waves

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Outlines

This lecture is a review of our works concerning the generation of solitons by resonance in nonlinear equations. Main subjects of the review

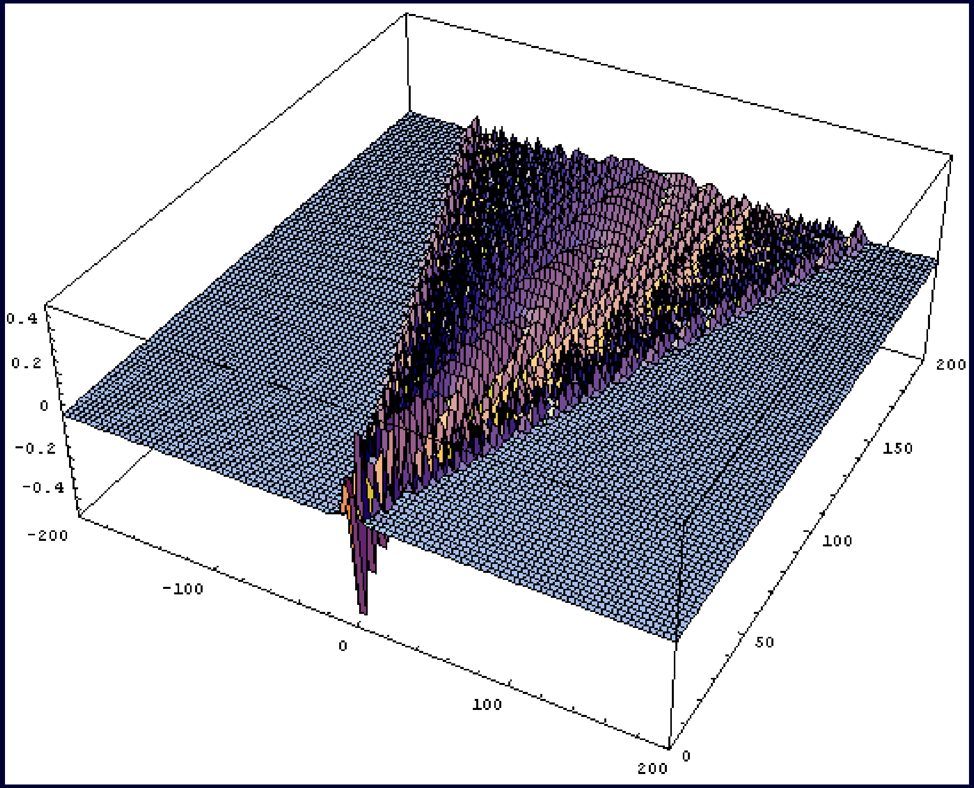
- perturbed nonlinear equations;
- weak resonances;
- generation of solitons as envelope function for packet of waves;
- connection formulas for parameters of packets and exciting force.

Let as consider the nonlinear Klein-Gordon equation with a cubic nonlinearity as an example

$$\partial_t^2 U - \partial_x^2 U + U + \gamma U^3 = \varepsilon^2 f(\varepsilon x) \exp\left\{ i \frac{S(\varepsilon^2 t, \varepsilon^2 x)}{\varepsilon^2} \right\} + \text{c.c.}.$$

Here $0 < \varepsilon \ll 1$, $\gamma = const$; $f(y)$ is smooth and rapidly vanishes as $y \to \pm\infty$. The function $S(y, z)$ and all derivatives are bounded.

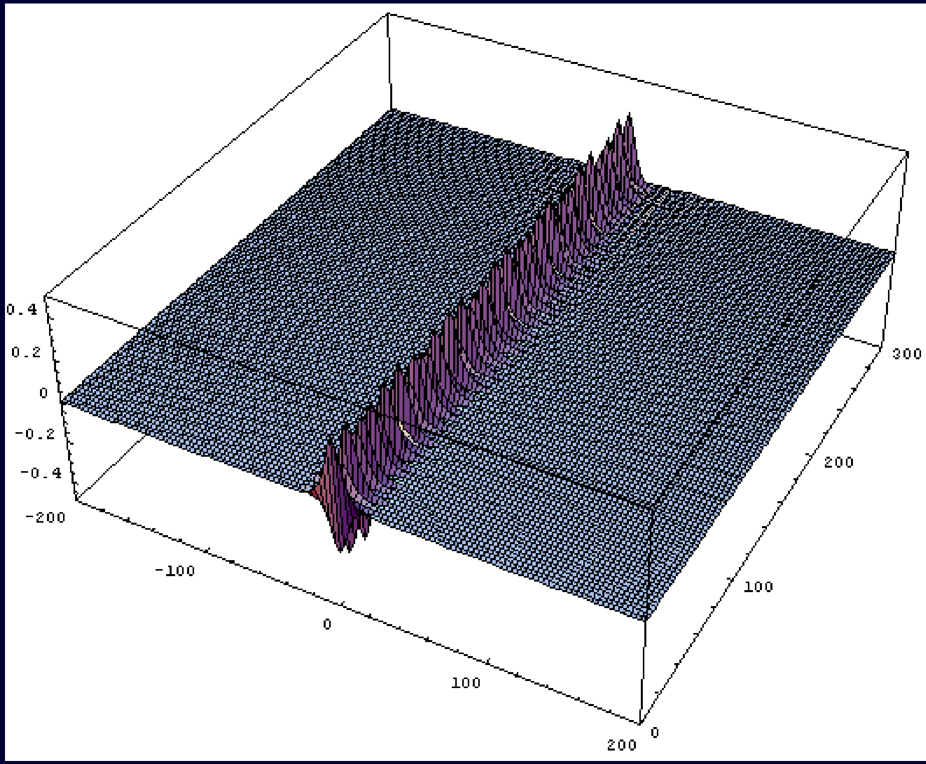
Typical solution of this equation has a form of wave packet. The wave packet propagate with distortion of envelope function in ordinary case.



Such distortion appears due to the dispersion, dissipation and nonlinearity. The dissipation control is not discussed here. The propagation of waves without distortion for the envelope is important for applications for exapmle in nonlinear optics.

The dispersion and nonlinearity oppose to each other. The dispersion tends to spread the wave packet and the nonlinearity tends to gather the packet. There exists a magic relation between typical scales of the wave packet in some special cases. The envelope of wave packet is a solution of the Nonlinear Schrödinger equation (NLSE). It was found by Kelley, Talanov and Zakharov in 1964-65 years. Later NLSE was integrated by inverse scattering transform method by Zakharov and Shabat in 1971.

The solitary packets of waves has a soliton of NLSE as the envelope function. Such wave packets propagate without of distortion on a large distance.



The solitary packets of waves would be more suitable for communication in optical fibers on a large distance if one can control the parameters of the envelope function for such packets.

Problems

- How to control the parameters of the envelope function for wave packets?
- How to obtain the wave packet with the given parameters?

There are some ways to obtain the solitary packet of waves.

- One of them is a spontaneous generation from an initial profile of the wave packet. Such method for the soliton generation used the asymptotic behaviour of the soliton equations. The solitonic envelope is formed as an asymptotic limit for the long time [Manakov, Ablowitz, 1973].

- Another way uses the transverse instability of the waves in the nonlinear medium [Kadomtsev and Petviashvily, 1973].

- Later L. Friedland and A. Shagalov (1998) obtained a numerical results for the autoresonant excitation of solitons.

Generation of solitons by weak resonance

Here we demonstrate a new approach to generate the packets of waves. We obtain this control due to small external perturbation. Our approach is based on slow passage of the perturbation force through the resonance. This way allows to generate the solitary packets of waves and effectively control their parameters.

The term "slow passage" through the resonance means passage through a local resonance. What does it mean?

Resonance

Consider the linear oscillator under periodic perturbation

$$\ddot{x} + 2\gamma\, \dot{x} + \omega_0^2 x = F_0 \cos \omega t,$$

where $F_0 =$ const is the amplitude of the perturbation and $\omega =$ const is the frequency of the perturbation.

The resonance phenomenon is well known. It is a standard phenomenon when the amplitude of the solution increases under the oscillating perturbation.

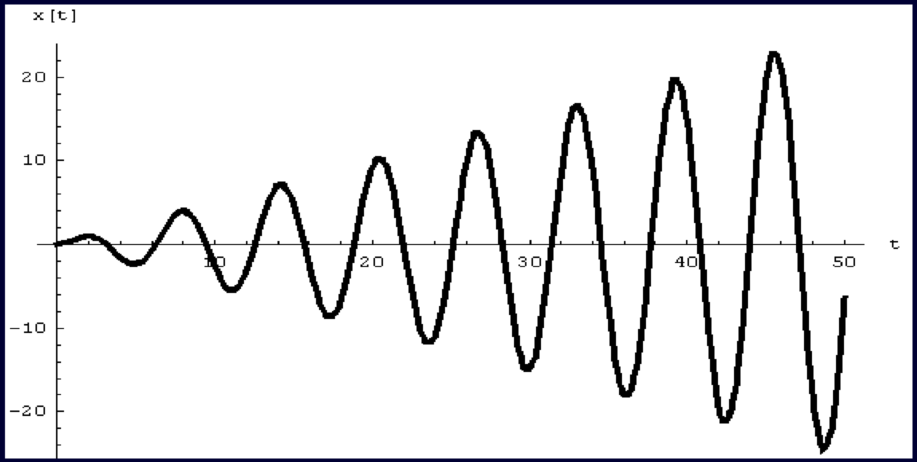


Figure 1: Resonant increase of the amplitude for $\gamma = 0, F_0 = 1, \omega = \omega_0 = 1.$

Let the frequency of the perturbation be $\omega = \omega(\varepsilon t)$ and there exists the moment t_0 such that $\omega_0 = \omega(\varepsilon t_0)$.

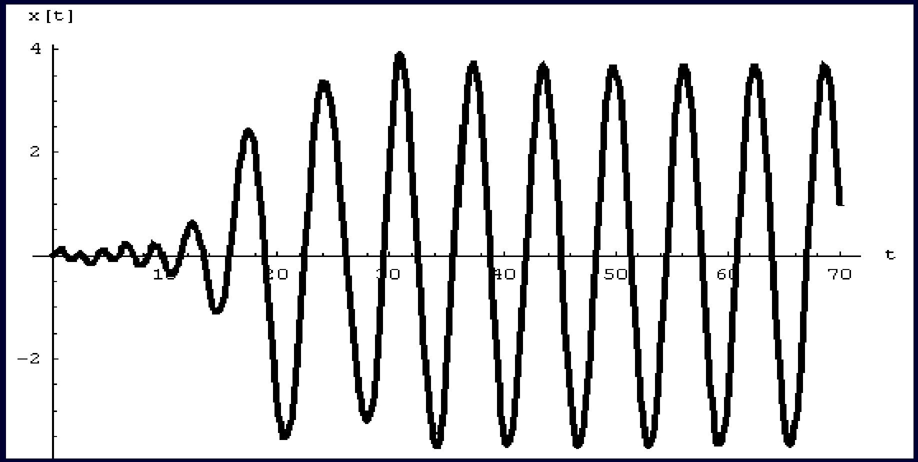


Figure 2: Increase of the amplitude due to local resonance for $\omega = -1 + \varepsilon t, \gamma = 0, F_0 = 1, \omega_0 = 1, \varepsilon = 0.1$

The resonance of this type is usually called **local** or **weak** resonance.

The main effect consist in appearance of the correction term of the order of $\sqrt{\varepsilon}$ for the asymptotic solution after the passage through the resonance domain.

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Generation of soliton on weak resonance

Let us consider the Klein-Gordon equation with a cubic nonlinearity

$$\partial_t^2 U - \partial_x^2 U + U + \gamma U^3 = \varepsilon^2 f(\varepsilon x) \exp\left\{ i\frac{S(\varepsilon^2 t, \varepsilon^2 x)}{\varepsilon^2} \right\} + \text{c.c.}. \quad (1)$$

Here $0 < \varepsilon \ll 1$, $\gamma = const$; $f(y)$ is smooth and rapidly vanishes as $y \to \pm\infty$. The function $S(y,z)$ and all derivatives are bounded. Denote: $x_j = \varepsilon^j x$, $t_j = \varepsilon^j t$, $j = 1, 2$; $l(t_2, x_2) \equiv (\partial_{t_2} S)^2 - (\partial_{x_2} S)^2 - 1$.

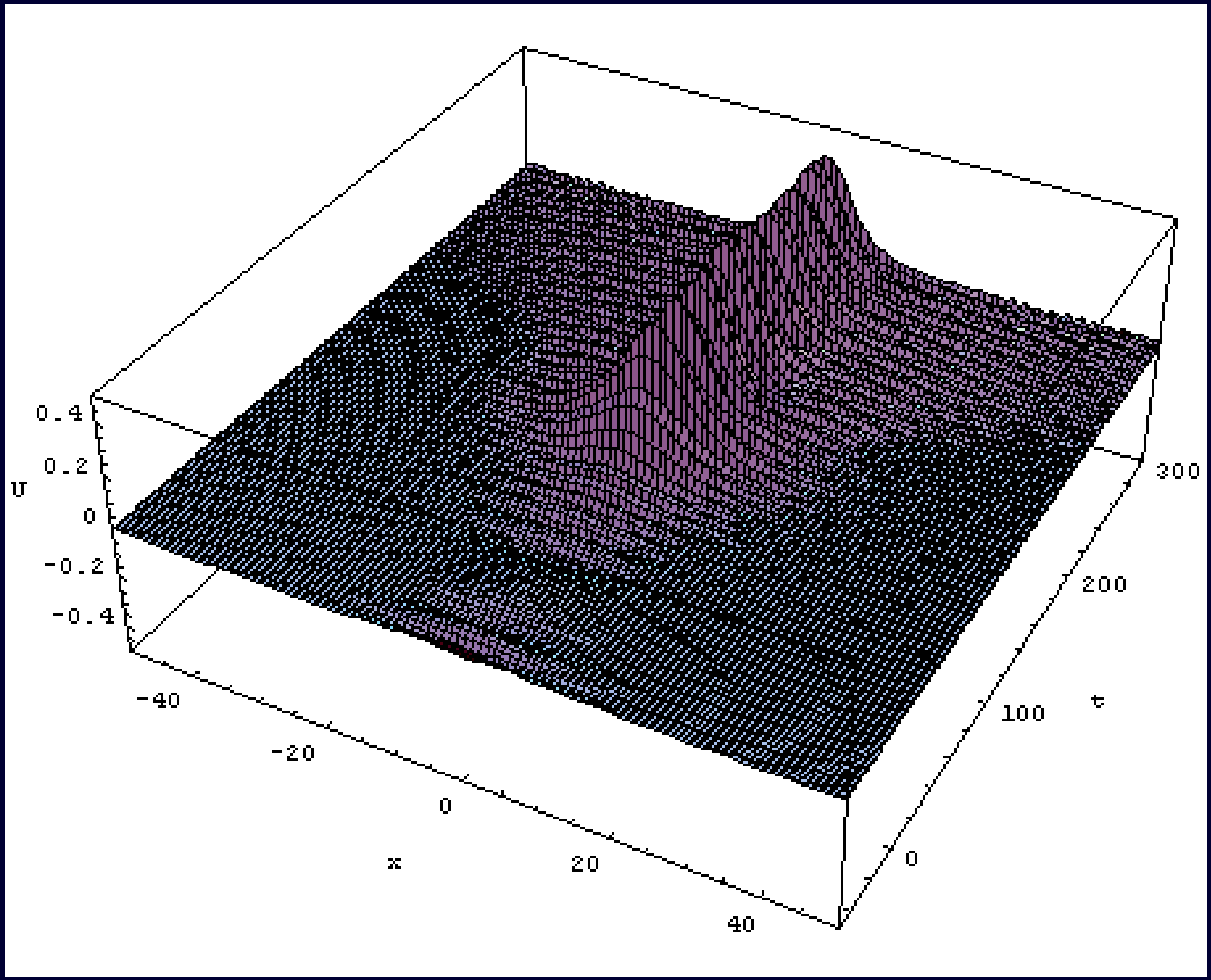
We will construct a special asymptotic solution of equation (1) such that:

$$U \sim -\varepsilon^2 \frac{f}{l} \exp(iS(t_2, x_2)/\varepsilon^2) + c.c.. \tag{2}$$

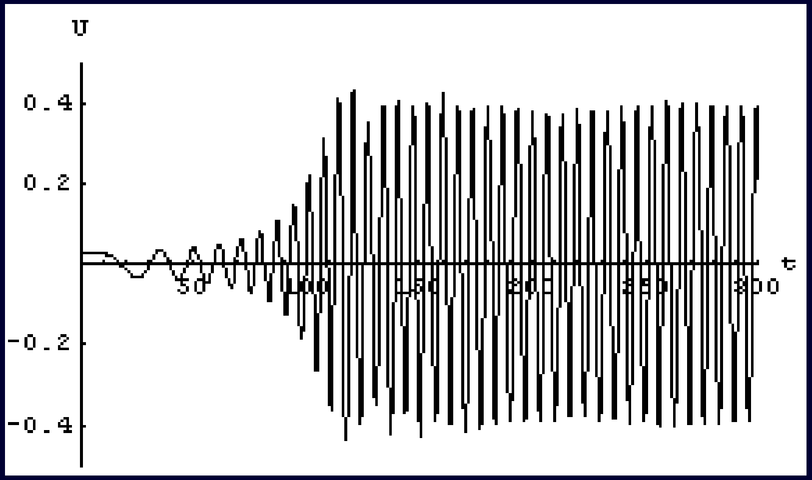
when $l < 0$.

Numeric simulations

In simplest case the generation of soliton looks like the follow picture

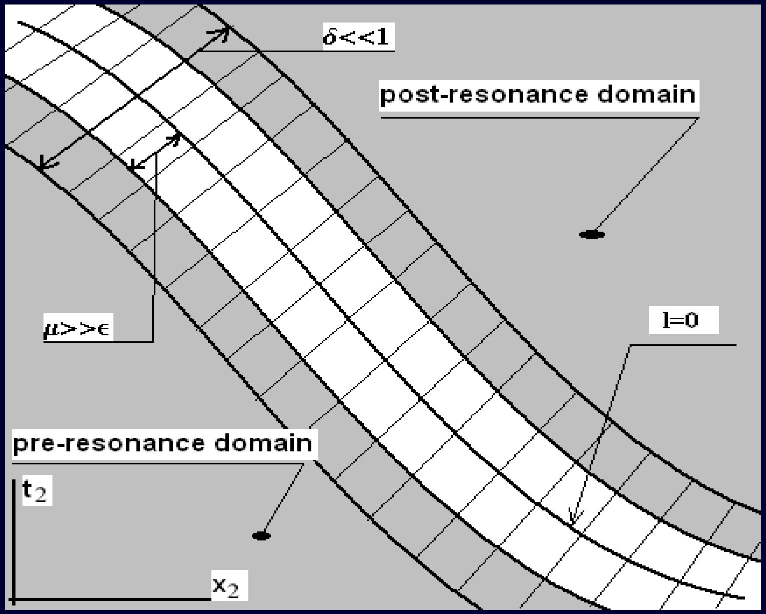

0.4
0.2
U
0
-0.2
-0.4
-40
-20
0
x
20
40
0
100
t
200
300

This picture shows the generation of the solitary packet of waves for equation (1) with special perturbation.



This picture shows a profile ($U(x,t)|_{x=0}$)of the packet.

Asymptotic analysis



All domains where we construct the solution is separated on three pairwise joint domains. The pre-resonant domain corresponds the forced oscillations with the amplitude of the order ε^2.

This oscillations break down when the driving force becomes resonant. The resonant layer is a thin domain near the resonant curve $l(x_2, t_2) = 0$. In this layer the amplitude of the oscillations increases up to the order ε. In the post-resonant domain the amplitude of the solution stabilizes on the order of ε.

Pre-resonant expansion

In the domain $-l \gg \varepsilon$ the formal asymptotic solution of equation (1) modulo $O(\varepsilon^{N+1})$ has the form

$$U = \sum_{n \geq 2}^{N} \varepsilon^n U_n(t, x, \varepsilon), \tag{3}$$

where

$$U_n = \sum_{k \in \Omega_n} U_{n,k}(t_2, x_2, \varepsilon x) \exp\left\{ ik \frac{S(t_2, x_2)}{\varepsilon^2} \right\}.$$

The set Ω_n for the higher-order term is described by the formula

$$\Omega_n = \begin{cases} \{\pm 1\}, & n \le 5; \\ \{\pm 1, \pm 3, \dots, \pm(2l+3)\}, & l = \big[(n-6)/4\big], \quad n \ge 6. \end{cases}$$

The functions $U_{n,k}$ and $U_{n,-k}$ are complex conjugated.

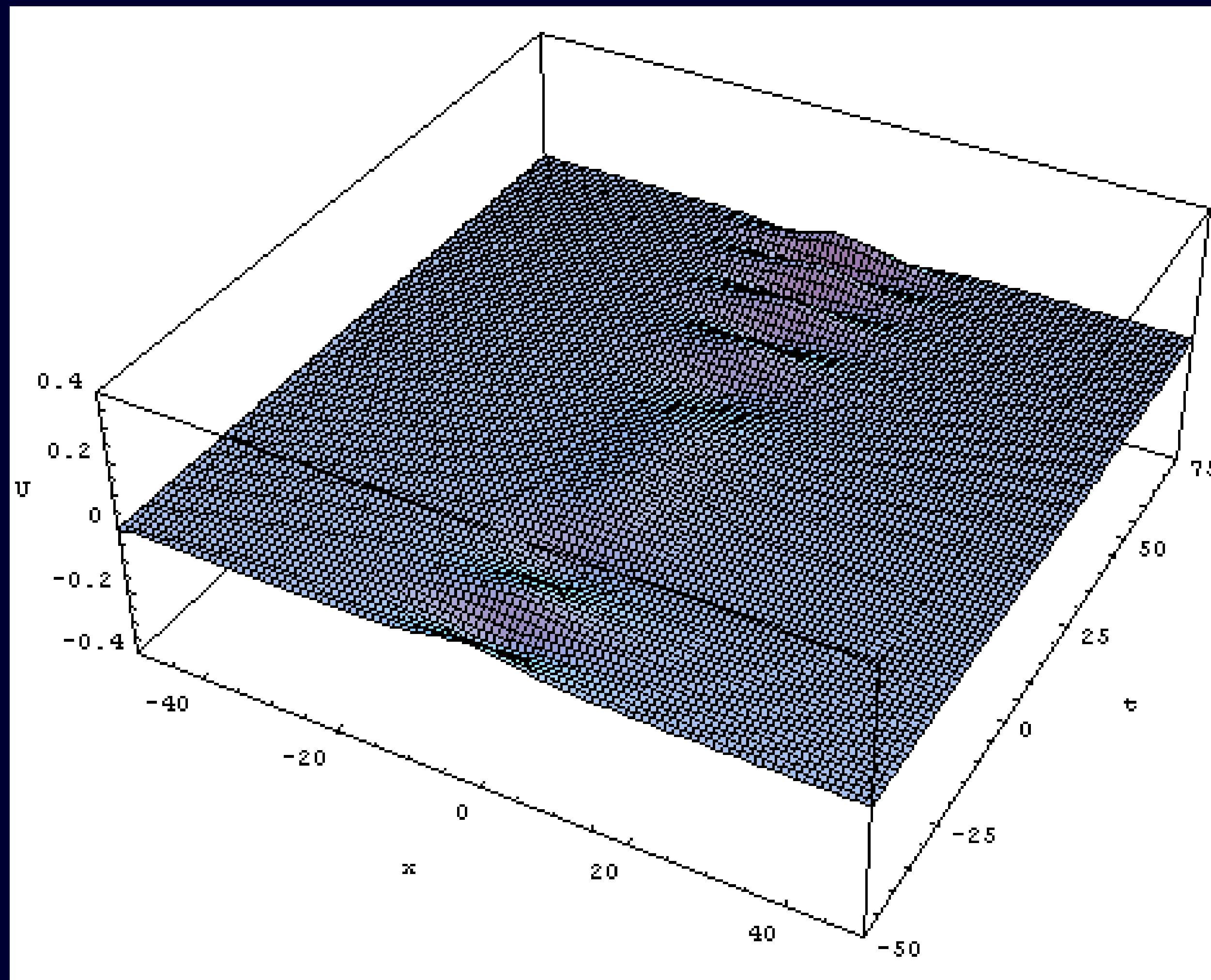
The coefficients of the asymptotics $U_{n,k}$ are defined out of algebraic equations

$$U_{2,1} = -\frac{f}{l}, \tag{4}$$

$$U_{3,1} = 2i\frac{\partial_{x_1} f \partial_{x_2} S}{l^2}, \tag{5}$$

$$U_{4,1} = \frac{2if[\partial_{t_2} S \partial_{t_2} l - \partial_{x_2} S \partial_{x_2} l] - 4(\partial_{x_2} S)^2 \partial_{x_1}^2 f}{l^3} - \frac{2i\partial_{t_2} f \partial_{t_2} S + \partial_{x_1}^2 f + i\partial_{t_2}^2 S f}{l^2}, \tag{6}$$

In this section we obtain the WKB-type of the asymptotic expansion which is valid before the resonance layer. This piece of the solution one can see on the following picture:


U
0.4
0.2
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-0.2
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-20
0
20
40
x
-50
-25
0
25
50
75
t

Resonant expansion

This part contains the asymptotic construction of the solution for equation (1) in the neighborhood of the curve $l = 0$. The domain of validity of this asymptotics intersects with the domain of validity of expansion (3). These expansions are matched.

In the domain $|l| \ll 1$ the formal asymptotic solution for equation (1) modulo $O(\varepsilon^{N+1})$ has the form

$$U = \sum_{n\geq 1}^{N} \varepsilon^n W_n(t_1, x_1, t_2, x_2, \varepsilon), \tag{7}$$

where

$$W_n = \sum_{k\in\Omega_n} W_{n,k}(x_2, t_2, x_1, t_1) \exp\left\{ik\frac{S(t_2, x_2)}{\varepsilon^2}\right\}, \tag{8}$$

The function $W_{n,1}$ is a solution of the problem for differential equations like the equation for the coefficient $W_{1,1}(x_1, t_1, x_2, t_2)$, which is defined by first order partial differential equation:

$$2i\partial_{t_2}S\partial_{t_1}W_{1,1} - 2i\partial_{x_2}S\partial_{x_1}W_{1,1} - \lambda W_{1,1} = f,$$

with a given asymptotic behaviour:

$$W_{1,1} \sim \frac{-f}{\lambda}, \quad \lambda \to -\infty.$$

Here $\lambda = l/\varepsilon$.

The asymptotic behaviour of $W_{1,1}$ as $\lambda \to \infty$ allows to relate the formulas (2) and (11).

The equation for $W_{1,1}$ may be written in the form of first order ordinary differential equation along the characteristic direction:

$$\frac{d}{d\sigma} W_{1,1} + \lambda W_{1,1} = f.$$

Such ordinary equation appears under studying of slowly passage through resonance for a one-dimensional oscillator with slowly varying frequency by Kevorkyan. The solution of equations of such type defines by Fresnel integrals.

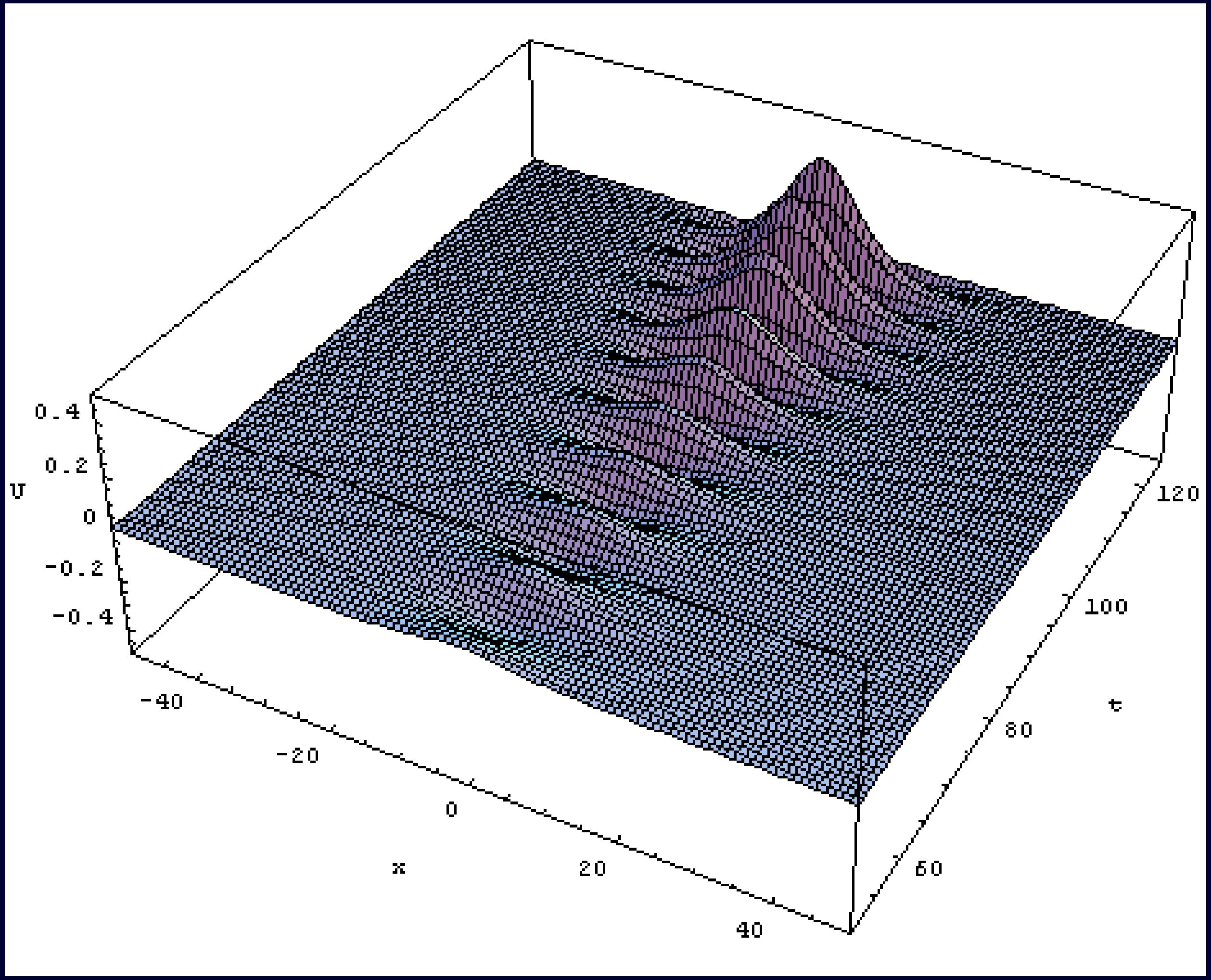
When $k \neq 1$ $W_{n,k}$ is the solution of algebraic equation. The functions $W_{n,k}$ and $W_{n,-k}$ are complex conjugated.

We obtain:

$$U(x,t,\varepsilon) \sim \varepsilon W_{1,1}(x_1,t_1,x_2,t_2)\exp\{iS/\varepsilon^2\} + c.c..$$

There is an essential difference between asymptotics (7) and external pre-resonance asymptotics (3). In the first place the leading-order term in (7) has an order ε while the leading order term in (3) has an order ε^2. In the second place the coefficients of asymptotics (7) depend on fast variables $x_1 = x_2/\varepsilon$ and $t_1 = t_2/\varepsilon$.

The resonant layer contains the strip where the solution increases due to the local resonance. This piece of the strip is shown on the following figure:


0.4
0.2
U
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-0.2
-0.4
-40
-20
0
x
20
40
60
80
t
100
120

Post-resonant expansion

In the domain $l \gg \varepsilon$ the formal asymptotic solution of equation (1) modulo $O(\varepsilon^{N+1})$ has a form

$$U(x,t,\varepsilon) = \sum_{1}^{N} \varepsilon^n \sum_{k=0}^{n-2} \ln^k(\varepsilon) \times$$
$$\times \bigg(\sum_{\pm\varphi} \exp\{\pm i\varphi(x_2,t_2)/\varepsilon^2\} \Psi_{n,k,\pm\varphi}(x_1,t_1,t_2) +$$
$$\sum_{\chi\in K'_{n,k}} \exp\{i\chi(x_2,t_2)/\varepsilon^2\} \Psi_{n,k,\chi}(x_1,t_1,t_2) \bigg). \qquad (9)$$

Here the function $\varphi(x_2, t_2)$ satisfies the eikonal equation

$$(\partial_{t_2}\varphi)^2 - (\partial_{x_2}\varphi)^2 - 1 = 0 \tag{10}$$

and initial condition on the curve $l = 0$:

$$\varphi|_{l=0} = S|_{l=0}, \quad \partial_{t_2}\varphi_{l=0} = \partial_{t_2}S|_{l=0}.$$

The leading-order term of the asymptotics is a solution of the Cauchy problem for the nonlinear Schrodinger equation

$$2i\partial_{t_2}\varphi\partial_{t_2}\Psi_{1,0,\varphi} + \partial_\xi^2\Psi_{1,0,\varphi} + i[\partial_{t_2}^2\varphi - \partial_{x_2}^2\varphi]\Psi_{1,0,\varphi} + \\ \gamma|\Psi_{1,0,\varphi}|^2\Psi_{1,0,\varphi} = 0,$$

$$\Psi_{1,0,\varphi}|_{l=0} = \int_{-\infty}^{\infty} df(x_1) \exp(i \int_0^d \chi l(x_1, t_1, \varepsilon)),$$

where ξ is defined by

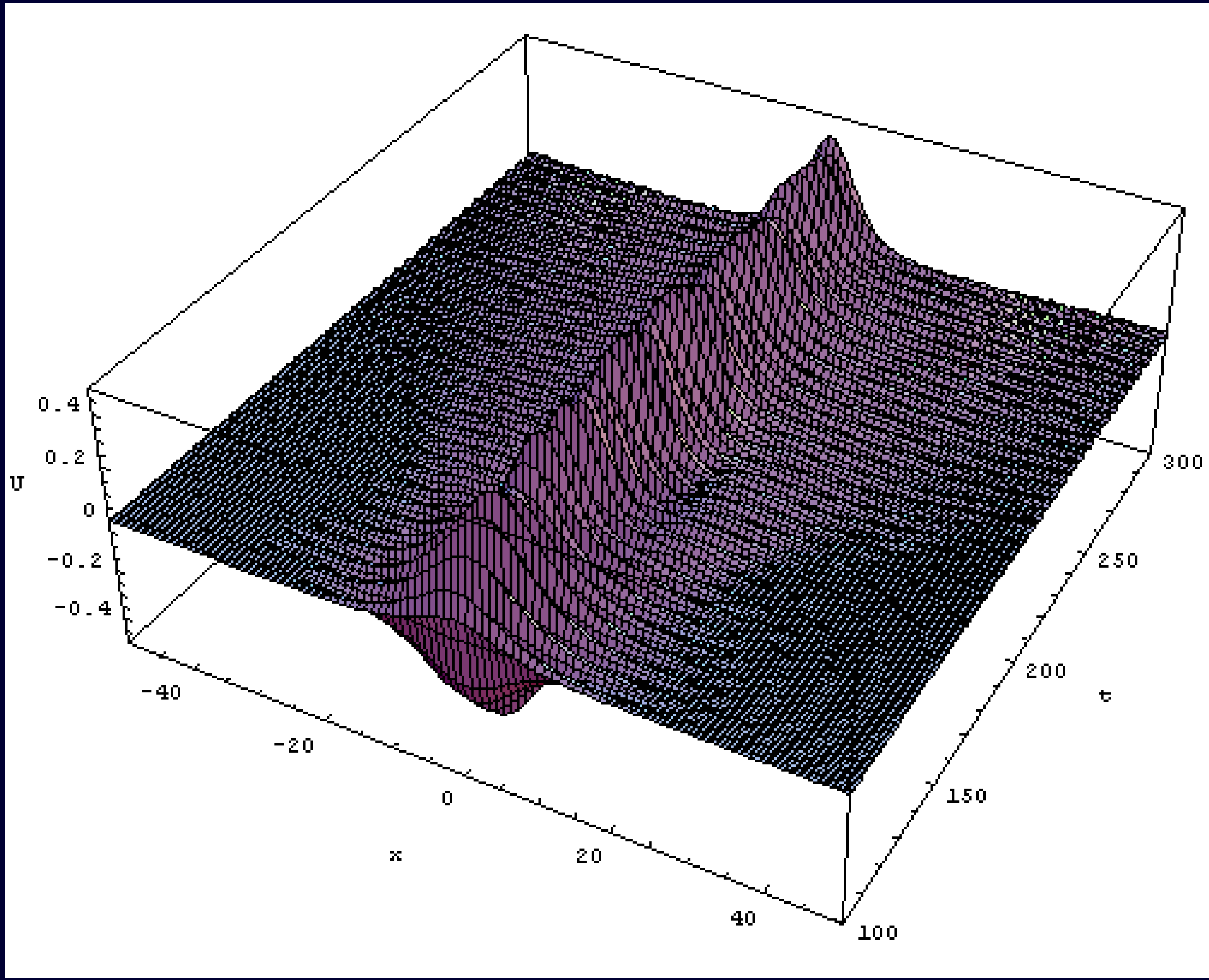
$$\frac{dx_1}{d\xi} = \partial_{t_2}\varphi, \quad \frac{dt_1}{d\xi} = \partial_{x_2}\varphi.$$

The coefficients $\Psi_{n,k,\pm\varphi}$ are determined from Cauchy problems for linearized Schródinger equation. The coefficients $\Psi_{n,k,\chi}$, $\chi \in K'_{n,k}$ are determined from algebraic equations. The set $K'_{n,k} = K_{n,k} \backslash \{\pm\varphi\}$. The phase set K_n for the $n-$th order term of the asymptotics as $l \to \infty$ is determined by formula

$$K_1 = \pm\varphi; \quad K_2 = \pm\varphi, \pm S,$$

$$K_n = \cup_{j_1+j_2+j_3=n} \chi_{j_1} + \chi_{j_2} + \chi_{j_3}, \quad , \chi_{j_k} \in K_{j_k}.$$

At last the post-resonant expansion has the following:



0.4
0.2
U
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-0.2
-0.4
-40
-20
0
x
20
40
100
150
200
t
250
300

Higher-order terms and matching

The structure of constructed asymptotic solution when $l < 0$ and $l > 0$ are sufficiently different. We concentrate on the description of the changing of the solution from the pre-resonant to post-resonant form. This transition takes place in the thin layer near the curve $l = 0$. In this transition layer the amplitude of the solution increases due to the resonant pumping. The value of the amplitude is defined by the width of the resonant layer. We found the width of the layer by construction and analysis of the higher-order terms of the asymptotic solution in all domains. This analysis looks very complicated but it is necessary to match the asymptotics of the solution in different domains and obtain formula (12). This formula defines the leading order term of the solution after the slowly passage through the resonance.

Main result

Let us formulate the main result of the work. If the solution of (1) has the form

$$U \sim -\varepsilon^2 \frac{f}{l} \exp(iS(t_2, x_2)/\varepsilon^2) + c.c.,$$

when $l < 0$, then in the domain $l > 0$ this asymptotic solution is

$$U(x,t,\varepsilon) \sim \varepsilon \Psi(x_1, t_1, t_2) \exp\{i\varphi(x_2, t_2)/\varepsilon^2\} + c.c. \qquad (11)$$

The phase function φ satisfies the eikonal equation

$$(\partial_{t_2}\varphi)^2 - (\partial_{x_2}\varphi)^2 - 1 = 0$$

with conditions

$$\varphi|_{l=0} = S|_{l=0}, \quad \partial_{t_2}\varphi|_{l=0} = \partial_{t_2}S|_{l=0}.$$

The envelope function of the leading-order term is a solution of the nonlinear Schródinger equation

$$2i\partial_{t_2}\varphi\partial_{t_2}\Psi + \partial_{\xi}^2\Psi + i[\partial_{t_2}^2\varphi - \partial_{x_2}^2\varphi]\Psi + \gamma|\Psi|^2\Psi = 0,$$

where the ξ is defined by

$$\frac{dx_1}{d\xi} = \partial_{t_2}\varphi, \quad \frac{dt_1}{d\xi} = \partial_{x_2}\varphi.$$

The initial condition for Ψ is

$$\Psi|_{l=0} = \int_{-\infty}^{\infty} df(x_1) \exp(i \int_0^d \mu l(x_1, t_1, \varepsilon)), \qquad (12)$$

The integration in this integral is done in the characteristic direction related with the equation for $W_{1,1}$.

Nonlinear Schrödinger equation (NLSE) is a mathematical model for wide class of wave phenomenons from the signal propagation in optical fiber to the surface wave propagation. This equation can be considered as an ideal model equation. Here we consider the NLSE perturbed by the small driving force.

$$i\partial_t\Psi + \partial_x^2\Psi + |\Psi|^2\Psi = \varepsilon^2 f e^{iS/\varepsilon^2}, \qquad 0 < \varepsilon \ll 1. \qquad (13)$$

Generation of soliton

$$i\partial_t \Psi + \partial_x^2 \Psi + |\Psi|^2 \Psi = \varepsilon^2 f \exp\{iS/\varepsilon^2\}, \quad 0 < \varepsilon \leq 1.$$

Generation of soliton

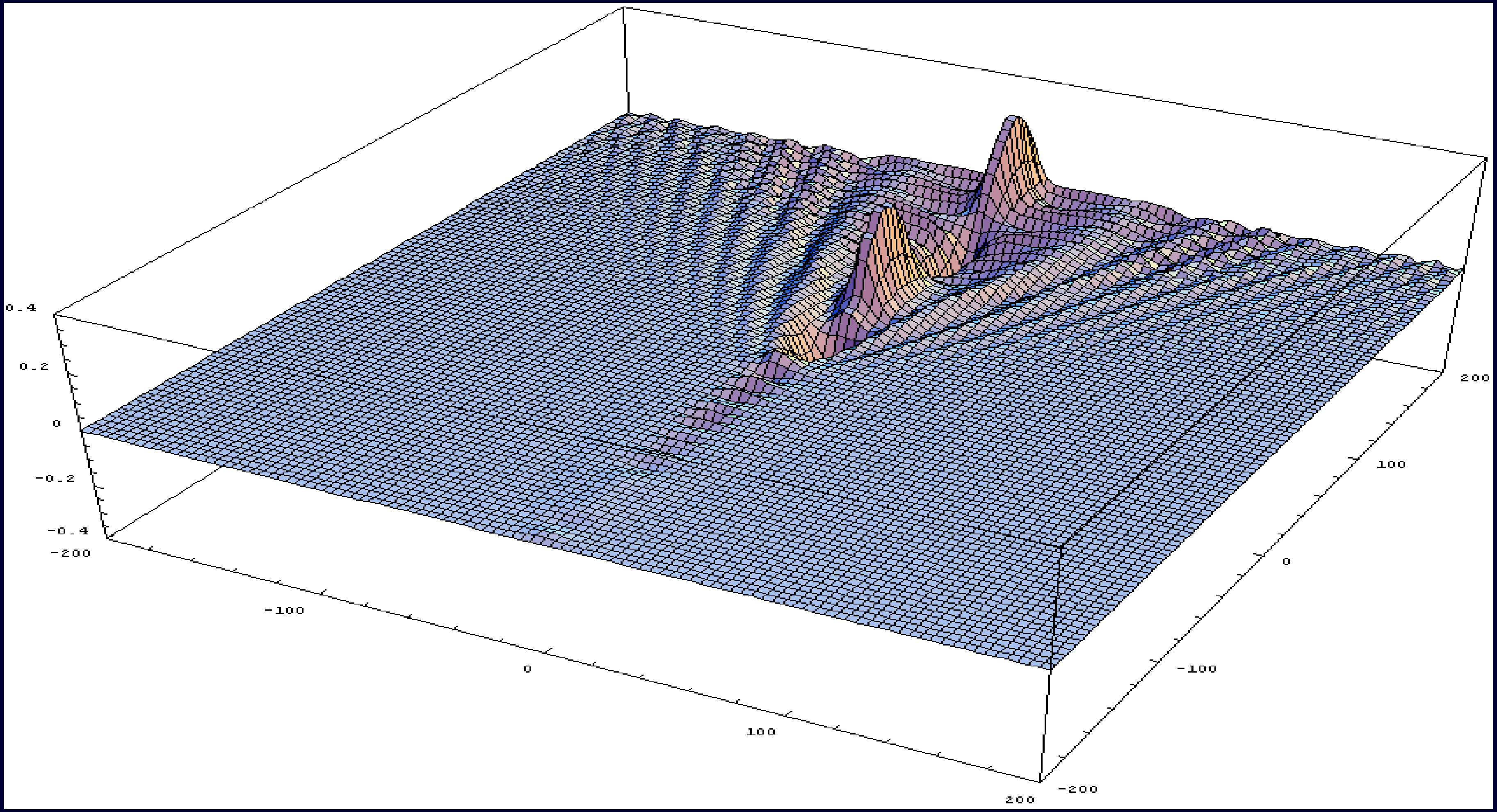
Theorem 1. *One phase asymptotic solution of the order of* $O(\varepsilon^2)$ *in the domain* $t \ll -\varepsilon^{-1}$ *will have the order of* $O(\varepsilon)$ *in the domain* $\varepsilon^{-1} \ll t \leq K\varepsilon^{-2},\ K = constant > 0$. *The leading-order term of the asymptotics*

$$\Psi = \varepsilon \overset{0}{u} + \mathcal{O}(\varepsilon^2)$$

is determined from the Cauchy problem for NLSE

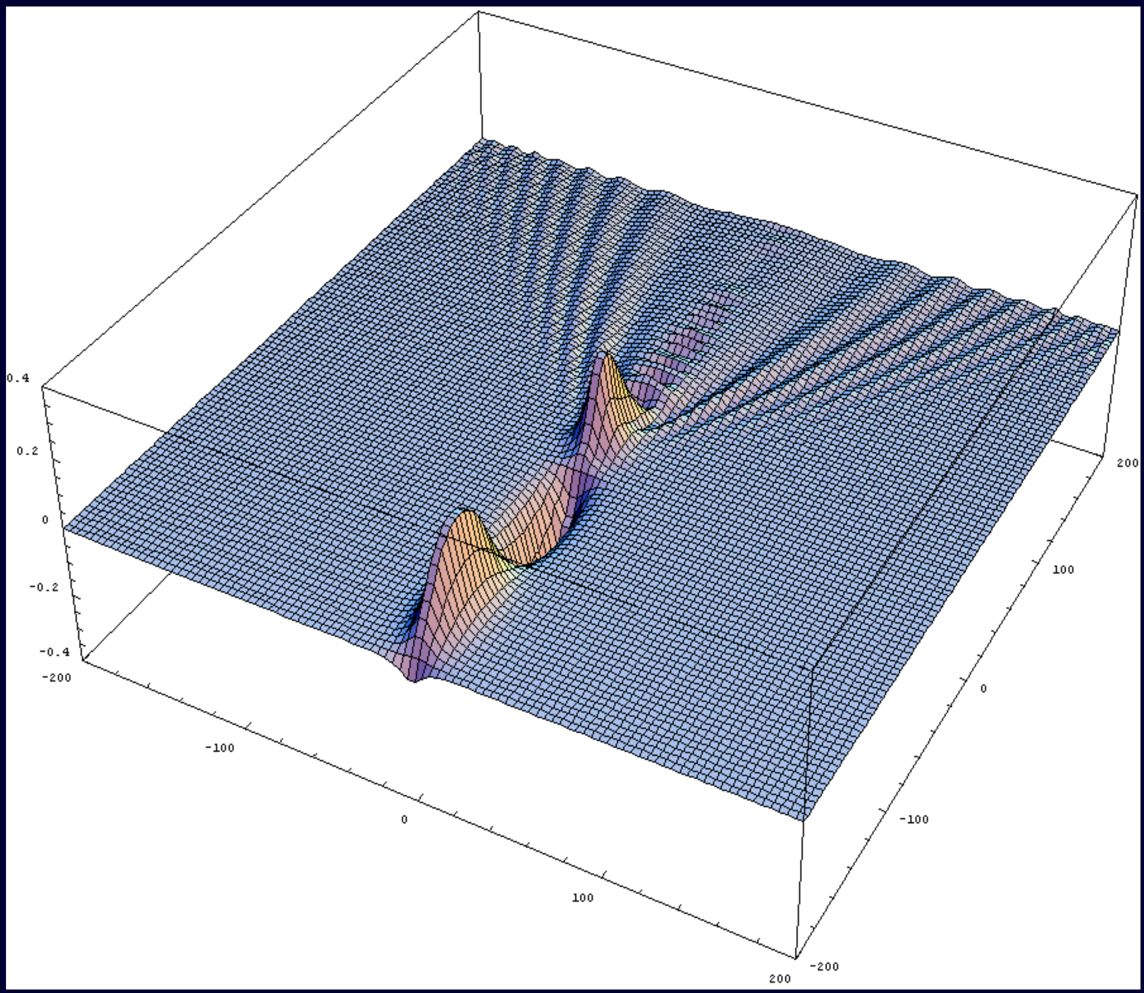
$$i \overset{0}{u}_{t_2} + \overset{0}{u}_{x_1 x_1} + | \overset{0}{u} |^2 \overset{0}{u} = 0, \quad \overset{0}{u} |_{t_2=0} = (1-i)\sqrt{\pi} f(x_1).$$

Generation of soliton



Annihilation of soliton

For numeric justification of our results we obtain the annihilate of the soliton on a local resonance.


0.4
0.2
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-0.2
-0.4
-200
-100
0
100
200
-200
-100
0
100
200

Finite amplitude waves

We consider the forced Boussinesq equation

$$U_{tt} - U_{xx} + a(U_x)^2_x + \varepsilon\gamma U_{xxxx} = \varepsilon^2 f(\varepsilon x)\exp\{iS(\varepsilon^2 x, \varepsilon^2 t)/\varepsilon^2\} + \text{c.c.}$$

We investigate the simplest case $S = S(\varepsilon^2 t) = (\varepsilon^2 t)^2/2$.

Resonance takes place on the curve $t = 0$.

References

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