Asymptotic behaviour of a solution for Kadomtsev-Petviashvili-2 equation*

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A main goal of this talk is to show how one can construct and justify the long time asymptotic behaviour of a decay solution for the Kadomtsev-Petviashvili-2 equation (KP):

$$\partial_x(\partial_t u + 6u\partial_x u + \partial_x^3 u) = -3\sigma^2\partial_y^2 u \quad \sigma^2 = 1. \tag{1}$$

The formal decay asymptotic at $t \to \infty$ for the KP-1 ($\sigma = -1$) equation was considered by Manakov, Santitni, Takhtadjyan (1980). However that work does not solve two important problems. The first one is an uniformity of the asymptotics with respect to all spatial variables. And the second one is a justification of the asymptotic solution. Here one must remind about one more work in which a generalized KP was studied. That work was done by N. Hayashi, P.Naumkin and J.-C.Saut (1999). Although in their paper the generalized KP equation was studied, but one can obtain the uniform asymptotic solution for the KP-1 by using their approach. However their approach does not permits to find the initial data for their asymptotic solution and the old problem about the justification of the asymptotic solution remains open.

In this talk the approach will being done by using two main points of rest. The first one is the inverse scattering method for the KP-2 equation which was developed by Ablowitz, Bar Yaacov, Fokas (1983). And the second one is the matching of the asymptotic expansions.

Inverse scattering transform (IST) for the KP-2

Let us denote an initial condition for equation KP-2 as: $u|_{t=0} = u_0(x,y)$. First step of the IST is solving a direct scattering problem for the function φ:

$$-\partial_y\varphi + \partial_x^2\varphi + 2ik\partial_x\varphi + u\varphi = 0, \quad \varphi|_{|k|\to\infty} = 1,$$

and scattering data are constructed by a formula:

$$F(k) = \frac{\mathrm{sgn}(-\mathrm{Re}(k))}{2\pi}\int_{\mathbb{R}^2} dxdy\, u_0(x,y)\varphi(x,y,k,0)\exp(\Omega),$$

where $\Omega = -i(k+\bar{k})x - (k^2 - \bar{k}^2)y$.

Statement of the problem

Next step is so-called $\bar{D}$-problem:

$$\begin{array}{r} \partial_{\bar{k}}\varphi = \psi\, F(-\bar{k})\exp(itS), \\ \partial_k\psi = -\varphi\, F(k)\exp(-itS); \end{array} \quad \begin{pmatrix}\varphi\\ \psi\end{pmatrix}|_{|k|\to\infty} = \begin{pmatrix}1\\ 1\end{pmatrix}, \quad (2)$$

where $S = 4(k^3 + \bar{k}^3) + (k+\bar{k})\xi - i(k^2 - \bar{k}^2)\eta$, $\xi = x/t$, $\eta = y/t$. Solving of this problem allows to obtain the functions φ and ψ at any time. At the final step one can obtain the solution of KP-2:

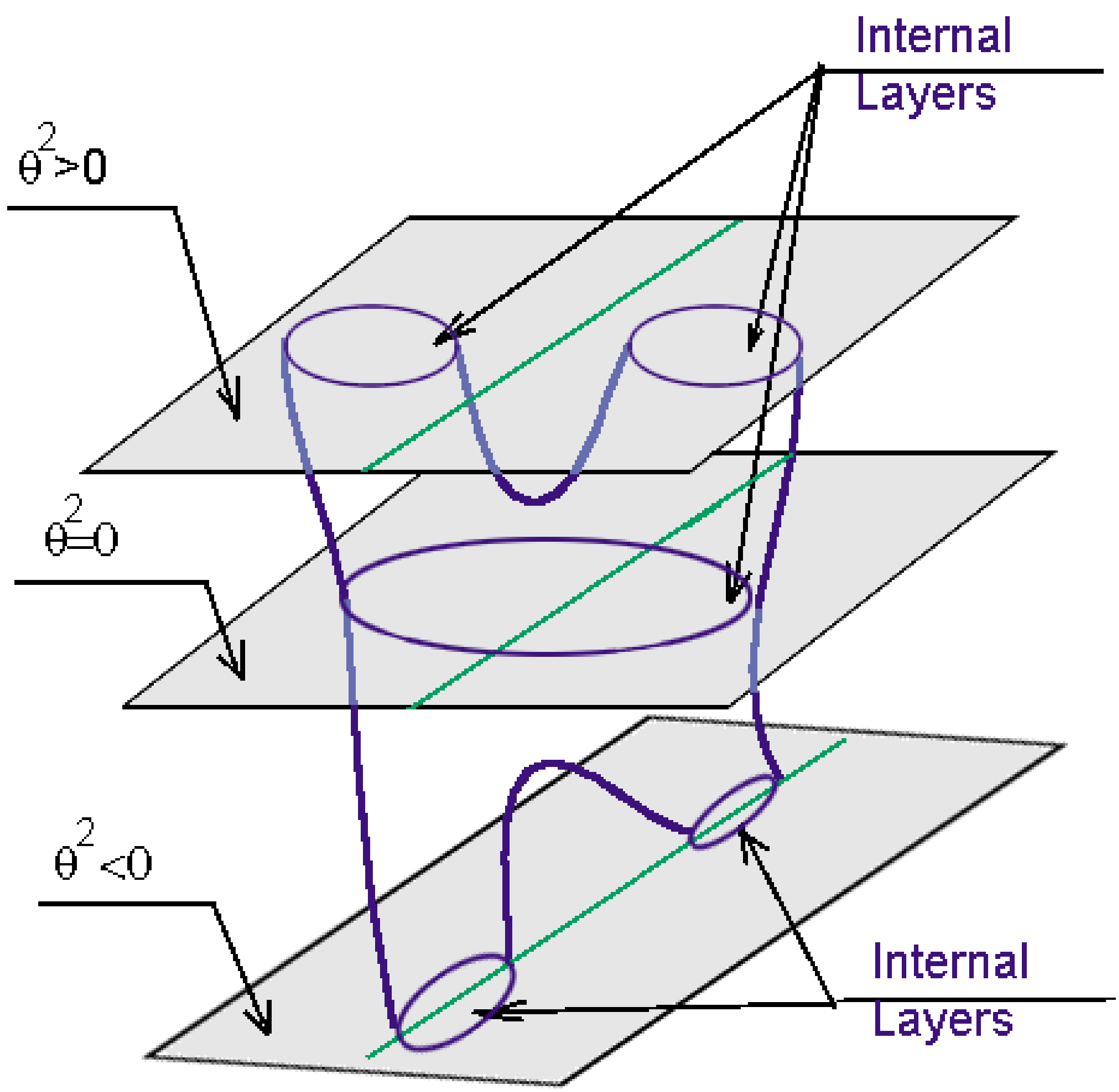
$$u(x,y,t) = \partial_x \int\int_{\mathbb{C}} dk\wedge d\bar{k}\, F(k)\psi(k,x,y,t)\exp(itS). \quad (3)$$

Our goal is to find the asymptotic solution for the $\bar{D}$-problem and to use this asymptotics to obtain the asymptotic behaviour of $u(x,y,t)$. This way seems very simple, because it is easy to see the main terms of the asymptotics as $t\to\infty$ of the $\psi \sim 1$ and $\varphi \sim 1$. But to justify this assumption one must construct the corrections of the asymptotics and to study their asymptotic behaviour.

Asymptotic solution of the $\bar{D}$-problem The $\bar{D}$-problem is reduced to

$$\begin{aligned} \partial_{\bar{k}}\mu &= \nu\, F(k)\,\exp(itS), \\ \partial_k \nu &= -\mu\, F(-\bar{k})\,\exp(-itS); \end{aligned} \quad \begin{pmatrix} \mu \\ \nu \end{pmatrix} |_{|k|\to\infty} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (4)$$

We will obtain the asymptotic behaviour of the $\bar{D}$ problem with fast oscillating as $t \to \infty$ and discontinuous as $\mathrm{Re}(k) = 0$ coefficients . The disposition of the stationary points of the functions $S(k, \bar{k}, \xi, \eta)$ on k, $\bar{k}$ with respect to the break line $\mathrm{Re}(k) = 0$ plays the important role in our asymptotic constructions. There exists the parameter $\theta = \sqrt{-12\xi - \eta^2}$ which defines the types of the asymptotics.



The phase function S depends on two real parameters (ξ, η). When $\theta \neq 0$ the stationary points of S are simple and the asymptotic sequence is $t^{-n/2}, n = 1, 2, \ldots$. On the curve $12\xi + \eta^2 = 0$ the confluence of two stationary points of the function S occurs. In this case we have one confluent stationary point. The structure of the asymptotic expansion of solution of (4) is changed here. As $|12\xi + \eta^2| \ll 1$ the expansion is constructed on the powers of $t^{-n/3}$, $n = 0, 1, 2, \ldots$ as an asymptotic sequence.

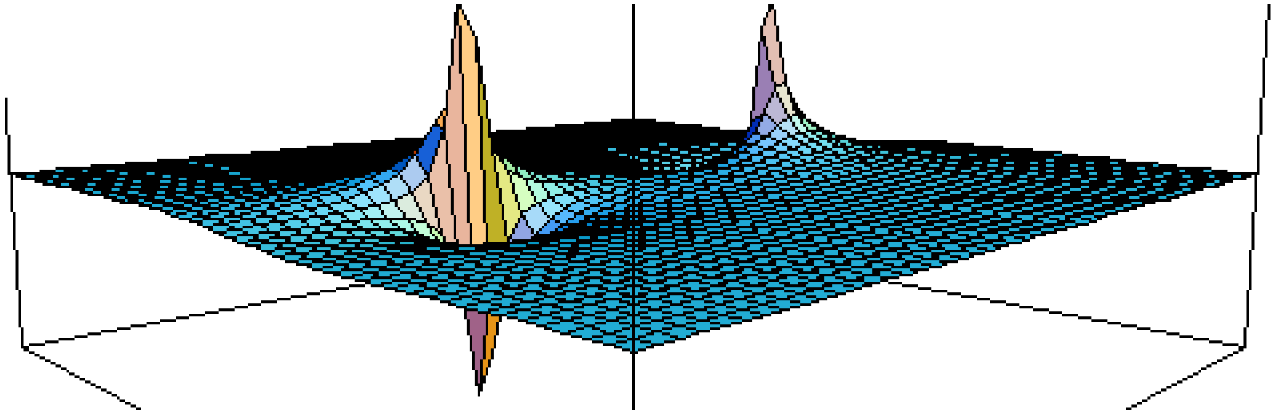
Let the system of the equations (4) have not the homogeneous solutions, $F(k) \in C^2 \cap L_1$ as $\mathrm{Re}(k) \neq 0$ and the parameters ξ and η satisfy the inequality $t^{1/3}|\theta|^2 \gg 1$, then:

when $\sqrt{t|\theta|}|k - k_{1,2}| \gg 1$ the formal asymptotic solution of the system (4) with respect to $mod(O(t^{-2}|\partial_k S|^{-3}))$ has the form:

$$\tilde{\mu} = 1 + t^{-1} \overset{1}{\mu} (k, \xi, \eta), \quad \tilde{\nu} = (t^{-1} \overset{1}{\nu}_1 (k, \xi, \eta) +$$

$$+ t^{-2} \overset{2}{\nu}_1 (k, \xi, \eta)) \exp(-itS) + t^{-1} \overset{1}{\nu}_0 (k, \xi, \eta);$$

$$\overset{1}{\mu} = \frac{-1}{2i\pi} \int\int_{\mathbb{C}} \frac{dp \wedge d\bar{p}}{k - p} \frac{f(-\bar{p}) f(p)}{i\partial_p S}, \quad \overset{1}{\nu}_1 = \frac{\mathrm{sgn}(\mathrm{Re}(k)) f(-\bar{k})}{i\partial_k S};$$



when $|\theta|^{-1}|k-k_j| \ll 1$ the formal asymptotic solution of the system (4) with respect to $mod(O(t^{-1}|\theta|^{-1}))$ has the form:

$$\tilde{\mu} = 1 + t^{-1} \overset{1}{M} (l_j, \xi, \eta),$$

$$\tilde{\nu} = \Big(t^{-1/2} \overset{1}{N} (l_j, \xi, \eta) + t^{-1} \overset{2}{N} (l_j, \xi, \eta)\Big) \exp(-itS),$$

where l_j , $j = 1, 2$, are defined by formula:

$$l_j = \sqrt{t}\Big(k - k_j\Big)\sqrt{\frac{\partial_k^2 S_j}{2} + 4(k-k_j)},$$

$$\partial_{l_j} \overset{1}{N}_j - 2il_j \overset{1}{N}_j = g_1, \quad \overset{1}{N}_j (l_j, \xi, \eta)|_{|l_j|\to\infty} = 0.$$

where $g_1 = -\sqrt{\frac{2}{\partial_k^2 S_j}}\text{sgn}(\text{Re}(k_j)) f(-\bar{k}_j)$. The solution of this boundary problem is done by the the Cauchy-Green formula:

$$\overset{1}{N}_j (l_j, \xi, \eta) = \text{sgn}(\text{Re}(k_j)) \frac{\sqrt{2} f(-\bar{k}_j)}{\sqrt{\partial_k^2 S_j}} \times$$

$$\times \frac{\exp(i(l_j^2 + \bar{l}_j^2))}{2i\pi} \int\int_{\mathbb{C}} \frac{dn \wedge d\bar{n}}{\overline{l_j - n}} \exp(-i(n^2 + \bar{n}^2))$$

The $\overset{1}{M}_j$ is the solution of the equation:

$$\partial_{\bar{l}_j} \overset{1}{M}_j = -\text{sgn}(\text{Re}(k_j)) f(k_j) \overset{1}{N}_j \sqrt{\frac{2}{\partial^2_{\bar{k}} S_j}}. \qquad (5)$$

The boundary condition for this equation is obtained from the matching with the external expansion.

$$\overset{1}{M}_j (l_j, \xi, \eta) = \overset{1}{M}_j^s (l_j, \xi, \eta) + C_j(\xi, \eta). \qquad (6)$$

The $\overset{1}{M}_j (l_j, \xi, \eta)$ is written as four-multiple integral:

$$\overset{1}{M}_j^s = \frac{-2f(k_j)f(-\bar{k}_j)}{|\partial^2_{\bar{k}} S_j|} \frac{1}{2i\pi} \int\int_{\mathbb{C}} \frac{dn \wedge d\bar{n}}{l_j - n} \frac{\exp(i(n^2 + \bar{n}^2))}{2i\pi} \times$$
$$\int\int_{\mathbb{C}} \frac{dm \wedge d\bar{m}}{\overline{n - m}} \exp(-i(m^2 + \bar{m}^2)).$$

When $\theta \to 0$ we need new scaling of the parameters θ and k. The scaled parameters are:

$$v = t^{1/3} \frac{\theta}{\sqrt{12}}, \quad p = t^{1/3}(k - k_0).$$

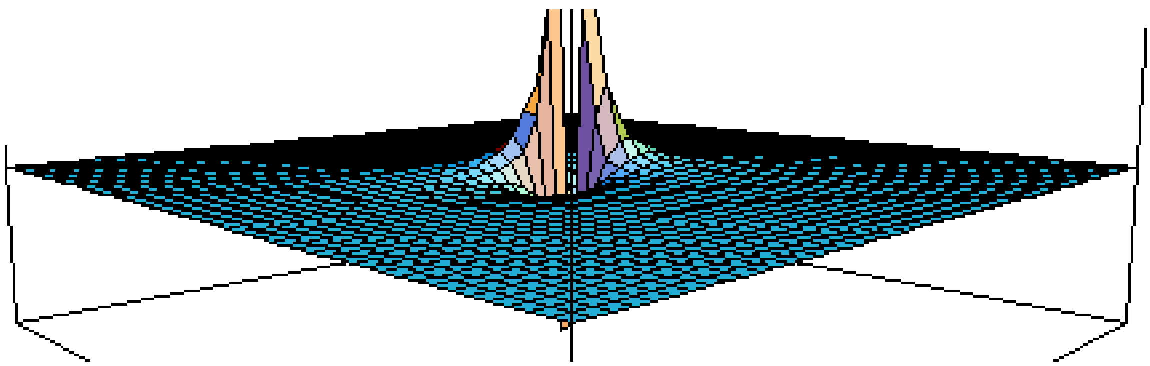
Let the system of the equations (4)be have no the homogeneous nontrivial solutions, $F(k) \in C^2 \cap L_1$ and the parameters ξ and η satisfy the inequality $|12\xi + \eta^2| \ll 1$, then:

when $|k - k_0| t^{1/3} \gg 1$ the formal asymptotic solution of the system (4) with respect to $mod(O(t^{-2/3}/|k - k_0|) + O(t^{-1}))$ has the form:

$$\tilde{\mu} = 1 + t^{-1} \overset{1}{m} (k, \xi, \eta),$$

$$\tilde{\nu} = t^{-2/3} \overset{1}{n}_0 + t^{-1}\Big(\overset{1}{n}_1 \exp(itS) + \overset{2}{n}_0 \Big)$$

$$\overset{1}{n}_0 = \frac{1}{\overline{\overline{k - k_0}}} f(-\bar{k}_0)[\phi^{+}_{00}(v^2) - \phi^{-}_{00}(v^2)],$$



$$\overset{1}{m} = \frac{1}{2i\pi} \int\int_{\mathbb{C}} \frac{dr \wedge d\bar{r}}{k-r} \frac{f(-\bar{r})f(r) - f(-\bar{k}_0)f(k_0)}{12i(r-k_0)^2} +$$
$$+ \frac{\overline{(k-k_0)} f(k_0) f(-\bar{k}_0)}{12i(k-k_0)^2} - \frac{1}{k-k_0}\Phi(v^2);$$

when $|k - k_0| \ll 1$ the asymptotic solution of the system (4)with respect to $mod(O(t^{-2/3}|k - k_0| + O(t^{-1}))$ has the form:

$$\tilde{\mu} = 1 + t^{-2/3} \overset{1}{\mathcal{M}} + t^{-1} \overset{2}{\mathcal{M}},$$

$$\tilde{\nu} = (t^{-1/3} \overset{1}{\mathcal{N}} + t^{-2/3} \overset{2}{\mathcal{N}} + t^{-1} \overset{3}{\mathcal{N}}) \exp(-itS);$$

where the corrections are defined by the problems:

$$\partial_p \overset{1}{\mathcal{N}} - i(12p^2 - v^2) \overset{1}{\mathcal{N}} = \mathrm{sgn}[\mathrm{Re}(-\bar{p})] f(-\bar{k}_0), \quad \overset{1}{\mathcal{N}} |_{|p|\to\infty} = 0.$$

$$\partial_{\bar{p}} \overset{1}{\mathcal{M}} = \mathrm{sgn}[-\mathrm{Re}(\bar{p})] f(k_0) \overset{1}{\mathcal{N}}, \quad \overset{1}{\mathcal{M}} |_{|p|\to\infty} = 0.$$

Justification of the asymptotic solution for $\bar{D}$-problem

Here we prove that the remainder of the asymptotics has order by $t^{-4/3}$ uniformly with respect to $k \in \mathbb{C}$ and this remainder has to be differentiable with respect to x. We call by the remainder of the asymptotics the difference between solution of the problem (4) and constructed asymptotic solutions when $\theta^2 t^{-2/3} \gg 1$, when $-\theta^2 t^{-2/3} \gg 1$ and when $|\theta| \ll 1$. The differentiability of the remainder will be important when we will construct an asymptotic behaviour of solution of the equation KP-2.

Theorem 1 *Let $\partial^\alpha f(k,\bar{k}) \in L_1 \cap C^1$ when $|\alpha| \leq 2$ when k is out of the imaginary axis and*

$$\sup_{z\in\mathbb{C}} \left| \int\int_{\mathbb{C}} \frac{dk \wedge d\bar{k}}{|k-z|} |F(k)| \right| < 2\pi,$$

then the solution of the problem (4) is:

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} \tilde{\mu} \\ \tilde{\nu} \end{pmatrix} + O(t^{4/3}), \qquad (7)$$

when $k \in \mathbb{C}$, $\xi, \eta \in \mathbb{R}$. The remainder of the asymptotics has to be differentiable with respect to x.

Asypmtotic behaviour of the solution for KP-2

To obtain the asymptotic of KP-2 we use he formula:

$$u(x,y,t) = \partial_x \int\int_{\mathbb{C}} dk \wedge d\bar{k}\, F(k)\psi(k,x,y,t) \exp(itS)$$

and asymptotic behaviour of the ψ.

Theorem 2 *Let* $(1+|k|)f \in L_1 \cap C$, $\partial^\alpha f \in L_1 \cap C$ *as* *Re*$(k) \neq 0$, $|a| \leq 2$ *and:*

$$\sup_{z\in\mathbb{C}} \int\int_{\mathbb{R}^2} d\kappa d\lambda \left|\frac{f(\kappa+i\lambda)}{\kappa+i\lambda-z}\right| < 2\pi, \qquad (8)$$

then the solution of the Cauchy problem of equation KP-2 for corresponding initial condition exists as $\forall t > 0$. *The asymptotic behaviour of the solution as* $t \to \infty$ *differs in different domains of variables* (x, y, t)*:*

as $-(12\xi+\eta^2)t^{1/3} \gg 1$*:*

$$u(x,y,t) = -4t^{-1}\frac{\pi}{12i\sqrt{-\eta^2-12\xi}} f\left(\frac{1}{2}\sqrt{-\eta^2-12\xi}+\frac{i\eta}{12}\right)\times$$
$$\times \exp\left(-11it\sqrt{-\frac{y^2}{t^2}-12\frac{x}{t}}\right) + c.c. + o(1).$$

as $(12\xi+\eta^2)t^{1/3} \gg 1$*:* $u = o(t^{-1})$;

as $|12\xi+12\eta^2| \ll 1$*:*

$$u(x,y,t) = 8it^{-1}\sqrt{\pi} f(i\eta/12)\Big(\textstyle\int_0^\infty dp_1\sqrt{p_1}\cos\left(8p_1^3 - zp_1\right) +$$
$$+\textstyle\int_0^\infty dp_1\sqrt{p_1}\sin\left(8p_1^3 - zp_1\right)\Big) + o(t^{-1}).$$

Here $\xi = x/t$, $\eta = y/t$, $z = 8\left(\frac{y^2}{12t^{4/3}} + \frac{x}{t^{1/3}}\right)$;

$$f(k) = \frac{1}{2\pi}\int\int_{\mathbb{R}^2} dx\,dy\,u_0(x,y)\varphi(x,y,k,0)\times$$
$$\times \exp(-i(k+\bar{k})x - (k^2-\bar{k}^2)y).$$

The domains of validity for the asymptotics of the solution of equation KP-2 are intersected and therefore they give combined asymptotics of the solution uniformly on plane of x, y.